MATHEMATICS MAGAZINE

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VOLUME 41 • MAY 1968 • NUMBER 3

MATHEMATICS MAGAZINE

ROY DUBISCH, *EDITOR*

EDITORIAL CORRESPONDENCE should be sent to the Editor-Elect, S. A. JENNINGS, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada. Articles should be typewritten and double-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

NOTICE OF CHANGE OF ADDRESS and other subscription correspondence should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Buffalo, New York, bi-monthly except July–August. Ordinary subscriptions are: 1 year \$4.00; 2 years \$7.00; 3 years \$10.00. Members of the Mathematical Association of America and of Mu Alpha Theta may subscribe at the special rate of 2 years for \$6.00. Single copies are 80¢.

Second class postage paid at Buffalo, New York and additional mailing offices.

ARE MANY 1-1-FUNCTIONS ON THE POSITIVE INTEGERS ONTO?

MARCEL F. NEUTS, Purdue University

Introduction. A simple "paradox" relating to the enumeration of the elements in a countable set may be described in the following way.

Every second a genie throws ten balls into an urn. The balls are numbered $1, 2, \cdots$ and at every throw he adds the next ten numbers to the urn so that at the nth throw the balls numbered $10n-9$, $10n-8$, $10n (n \geqq 1)$ are added. This goes on forever.

Another genie removes one ball from the urn after each addition, but he must guarantee that every ball will eventually be thrown out. If he can see the balls, there is of course no problem. He can remove the balls $1, 2, 3, \cdots$ successively and for any natural number k, he knows when it enters the urn and when it is removed. It enters the urn at the $[k/10]+1$st throw and is removed after the kth throw. Though the number of balls in the urn tends to infinity, any given ball is eventually thrown out. No ball stays in the urn forever. This is one of the paradoxes of infinity, stated by Georg Cantor and discussed as the "Tristam Shandy paradox" by Russell [7].

For every k, the length of time T_k spent in the urn by the ball k is given by

$$T_k = k - 1 - \left[\frac{k}{10}\right], \qquad k = 1, 2, \cdots.$$

There are, of course, many more rules which will guarantee the eventual removal of every given ball. Clearly, there are also rules which will leave one or more, even infinitely many balls in the urn. Thus if he removed successively the balls $10, 20, 30, \cdots$ all numbers which are not multiples of ten would stay in forever.

To compound the sad fate of the second genie, we assume next that he cannot see the numbers on the balls and that the balls are, in fact, completely indistinguishable. The problem is now, whether or not there is a way in which the second genie can remove every ball from the urn. Or, to state the "paradox": does the ability of the second genie to enumerate all the balls depend on the enumeration already given?"

We must still describe a rule, but one that does not depend on the numbering of the balls at all. The first such procedure that comes to mind is to draw at each removal the ball *at random from among those still in the urn.* This rule is appealing, because every ball in the urn at every drawing is given the same chance of being removed. Before the nth removal there are $9n+1$ balls in the urn. We assume, that, independent of the past, any one of these balls has a probability $(9n+1)^{-1}$ of being taken out.

This rule will be satisfactory for Genie II, if we can show that, with probability one, every given ball is eventually removed from the urn.

Since the balls are completely indistinguishable, the genie must rely on chance and a chance procedure with the stated property is the best one can wish for.

We will prove below that "random removal" has this property but first we leave the world of fairy tales and formulate a more general mathematical problem.

Mathematical formulation. Let $a_1 < a_2 < \cdots$ be a strictly increasing sequence of positive integers and let $\mathfrak{F}$ be the family of all functions from the positive integers into the positive integers which satisfy

$$
\begin{aligned}
f(n) &\leqq a_n, \qquad n \geqq 1, \\
f(n) &\neq f(\nu), \qquad \nu \neq n.
\end{aligned} \tag{1}
$$

On the class of all subsets of $\mathfrak{F}$, we can define probabilities satisfying

$$
\begin{aligned}
P\{f(1) = k\} &= \frac{1}{a_1}, \qquad 1 \leqq k \leqq a_1 \\
&= 0, \qquad k > a_1
\end{aligned} \tag{2}
$$

and, for all $n > 1$,

$$
P\{f(n) = k \mid f(1), \cdots, f(n-1)\} = \frac{1}{a_n - n + 1} \tag{3}
$$

$$
1 \leqq k \leqq a_n,\ k \neq f(\nu) \quad \nu = 1, \cdots, n-1,
$$

and zero elsewhere.

This assignment of probabilities corresponds to the following scheme: for every $n \geqq 1$, the value of $f(n)$ is chosen at random from among the numbers $1, 2, \cdots, a_n$ which have not been chosen previously. That the requirements (2) and (3) determine a unique probability measure on the class of all subsets of $\mathfrak{F}$ may be proved from first principles or by appealing to the general theorem 8.3.A, p. 137 in Loève [**3**]. The uniqueness of the probability measure P also follows from property 1 below and the classical extension theorem for measures.

This assignment of probabilities corresponds to the requirement which, loosely stated, says that all functions in $\mathfrak{F}$ are "equally probable." To see this we prove

PROPERTY 1.

$$
P\{f(1) = \alpha_1, \cdots, f(m) = \alpha_m\} = [a_1(a_2 - 1) \cdots (a_m - m + 1)]^{-1}, \tag{4}
$$

is $\alpha_i \leqq a_i$, for $i = 1, \cdots, m$ and no two α_i's are equal. For all other m-tuples $(\alpha_1, \cdots, \alpha_m)$, this probability is zero.

Proof. Use the chain rule of conditional probability; then

$$
\begin{aligned}
&P\{f(1) = \alpha_1, \cdots, f(m) = \alpha_m\} = \\
&P\{f(1) = \alpha_1\} P\{f(2) = \alpha_2 \mid f(1) = \alpha_1\} \cdots \\
&\cdots P\{f(m) = \alpha_m \mid f(1) = \alpha_1, \cdots, f(m-1) = \alpha_{m-1}\},
\end{aligned}
$$

which yields (4) upon substitution.

Remarks. The space of functions $\mathfrak{F}$ with the probability assignment $P(\cdot)$ may be identified with the following urn scheme. Suppose that the urn contains initially a_1 balls, numbered $1, \cdots, a_1$. One ball is drawn out and new balls, numbered $a_1+1, \cdots, a_2$ are added. Again a ball is drawn out at random and removed, and balls, numbered $a_2+1, \cdots, a_3$ are added and so on. If we denote by X_n the number of the nth ball drawn, then the sequence $\{X_1, X_2, \cdots\}$ defines a function in $\mathfrak{F}$. We see that the sequence $a_1, a_2, \cdots$ characterizes the set $\mathfrak{F}$ and the probability assignment $P(\cdot)$. The scheme, discussed in the Introduction, corresponds to $a_n = 10n$.

Let the event that $X_n = k$ be denoted by $\{X_n = k\}$; then $\bigcup_{n=1}^{\infty}\{X_n = k\} = B_k$ is the event that for some n the number k is drawn at the nth drawing. Since the events $\{X_n = k\}$ are disjoint, we have

$$P(B_k) = \sum_{n=1}^{\infty} P\{X_n = k\}. \tag{5}$$

We are interested in conditions on the sequence $\{a_n\}$ under which

$$\forall k: P(B_k) = 1. \tag{6}$$

THEOREM 1. (a) *If* $P(B_{k_0}) = 1$ *for some* $k_0 \geqq 1$, *then* (6) *holds.* (b) *Property* (6) *holds if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{a_n - n + 1} = \infty. \tag{7}$$

Proof. Let k_0 be a positive integer and $n^* = \min\{n: a_n \geqq k_0\}$; then

$$P(B_{k_0}^c) = P\left[\bigcap_{n=n^*}^{\infty} (X_n \neq k_0)\right] = \prod_{n=n^*}^{\infty}\left(1 - \frac{1}{a_n - n + 1}\right), \tag{8}$$

so that $P(B_{k_0}^c) = 0$ if and only if the infinite product diverges, or equivalently if the sum (7) does.

However, the divergence of this sum is independent of the value of k_0, which proves part (a).

COROLLARY. *If* (7) *holds, then for any nonvoid set of indices* $\{k_1, k_2, \cdots\}$ *we have*

$$P\left\{\bigcap_{i=1}^{\infty} B_{k_i}\right\} = 1. \tag{9}$$

Proof.

$$P\left\{\bigcap_{i=1}^{\infty} B_{k_i}\right\} = 1 - P\left\{\bigcup_{i=1}^{\infty} B_{k_i}^c\right\}$$

but

$$0 \leqq P\left\{\bigcup_{i=1}^{\infty} B_{k_i}^c\right\} \leqq \sum_{i=1}^{\infty} P(B_{k_i}^c) = 0$$

by Theorem 1.

Remark. The corollary says that, with probability 1, all positive integers appear in an infinite sequence of drawings in an urn corresponding to a sequence $\{a_n\}$ which satisfies (7). We can therefore say that if and only if condition (7) is satisfied "almost all functions in the class $\mathfrak{F}$ are onto."

An example of a class of functions which do not satisfy condition (7). It is, of course, easy to give examples of such classes of functions, just by choosing a_n a fast growing sequence. The following example is of some particular interest as it relates to a familiar proof of the countability of the set of all rational numbers.

Let E_n be the set of all rational numbers in (0, 1) which can be written as irreducible fractions with denominator at most equal to n. The number of elements in E_n is given by:

$$a_n = \sum_{\nu=2}^{n} \phi(\nu) \; n \geqq 2. \tag{10}$$

Set $a_1 = 1$. $\phi(\nu)$ is Euler's ϕ-function, i.e., $\phi(\nu)$ is the number of integers a, with $1 \leqq a \leqq \nu$ which are relatively prime to ν.

Therefore, for $n \geqq 2$, we have

$$\frac{1}{a_n - n + 1} = \left[\sum_{\nu=2}^{\infty} \phi(\nu) - (n-1) \right]^{-1}. \tag{11}$$

However, it is known that

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{\nu=1}^{n} \phi(\nu) = \frac{3}{\pi^2} \cdot \tag{12}$$

[See **1**, Vol. 3, p. 172, formula (32).]

Therefore

$$\frac{1}{a_n - n + 1} \sim \frac{1}{n^2} \cdot \frac{\pi^2}{3}, \tag{13}$$

so that the series in (7) *converges*.

Remark. An interesting problem is to find an expression for the probability that a function in $\mathfrak{F}$ is onto if condition (7) is not satisfied.

Functions of at most linear growth. The class $\mathfrak{F}$ of functions corresponding to

$$a_n = a + b(n-1) \qquad a \geqq 1, b \geqq 1, n \geqq 1, \tag{14}$$

is of particular interest.

Since $a_n - n + 1 = a + (b-1)(n-1)$, the series in (7) diverges. Consider any ball in the urn just before the nth drawing and let T be the additional number of drawings required before this ball is removed, then:

$$P\{T > \nu\} = \prod_{\alpha=n}^{n+\nu} \left[1 - \frac{1}{a + (b-1)(\alpha - 1)} \right]$$

$$(15)\qquad \begin{aligned} &= \prod_{\alpha=n}^{n+\nu}\left(\frac{a-b}{b-1}+\alpha\right)\left(\frac{a-b+1}{b-1}+\alpha\right)^{-1} \\ &= \frac{\Gamma\left(\frac{a-b}{b-1}+n+\nu+1\right)\Gamma\left(\frac{a-b+1}{b-1}+n\right)}{\Gamma\left(\frac{a-b}{b-1}+n\right)\Gamma\left(\frac{a-b+1}{b-1}+n+\nu+1\right)} \\ &= \frac{B\left[\frac{a-b}{b-1}+n+\nu+1,\ \frac{1}{b-1}\right]}{B\left[\frac{a-b}{b-1}+n,\ \frac{1}{b-1}\right]}, \qquad b>1.\ \nu \geqq 0 \end{aligned}$$

in terms of Euler's gamma and beta functions [1]. The case $b=1$ is trivial and leads to a geometric distribution for T. The expected value of the random variable T is given by $(b>1)$

$$(16)\qquad \begin{aligned} E(T) &= \sum_{\nu=0}^{\infty} P[T>\nu] = \frac{1}{B\left[\frac{a-b}{b-1}+n,\ \frac{1}{b-1}\right]} \\ &\quad \cdot \sum_{\nu=0}^{\infty}\int_0^1 u^{(a-b/b-1)+n+\nu}\,(1-u)^{(1/b-1)-1}du \\ &= \frac{1}{B\left[\frac{a-b}{b-1}+n,\ \frac{1}{b-1}\right]}\int_0^1 u^{(a-b/b-1)+n}\,(1-u)^{(1/b-1)-2}du \end{aligned}$$

since the integral on the right diverges.

This leads to the observation that though the ball in the urn at time n will be drawn out eventually with probability 1, the *expected number of drawings required is infinite.*

To illustrate the enormous growth of waiting times in terms of n, we consider an extremely simple case of (14) and appeal to some results which were proved in the theory of record observations.

Let $a=2$ and $b=2$ so that the number of balls in the urn at the nth drawing is $n+1$ $(n\geqq 1)$.

Consider the following process. Before the first drawing, mark one of the two balls and continue drawing until the marked ball is drawn. When this happens, mark one of the balls in the urn just before the next drawing and continue drawing until this ball is drawn. When this happens, mark again one of the balls in the urn and so on.

It is easy to see that by this procedure, we generate a sequence of independent Bernoulli trials in which the probability of success at the nth trial is $1/(n+1)$. Success is defined as the drawing of a previously marked ball.

Suppose now that we define the random variable L_m as the total number of drawings required until the mth marked ball is drawn out. Equivalently L_m is the number of trials until the mth success in a sequence of independent Bernoulli trials in which the probability of success at the nth trial is $p_n = 1/(n+1)$.

The random variable L_m was studied by Foster and Stuart [4] and by Alfred Rényi [6] in connection with the study of recordbreaking observations. They proved among other things that

$$(L_m)^{1/m} \to e \tag{17}$$

with probability 1 and that

$$P\{\log L_m \leqq m + t\sqrt{m}\} \to \int_{-\infty}^{t} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}, \tag{18}$$

so that the limiting distribution of $(\log L_m - m)/\sqrt{m}$ is a unit normal distribution.

However if we set $\Delta_m = L_m - L_{m-1}$, $m \geqq 1$, $L_0 = 0$, then Neuts [5] has shown that

$$(\Delta_m)^{1/m} \to e \text{ in probability} \tag{19}$$

and

$$P\{\log \Delta_m \leqq m + t\sqrt{m}\} \to \int_{-\infty}^{t} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}, \tag{20}$$

so that the limiting behavior of $L_m = \Delta_1 + \Delta_2 + \cdots + \Delta_m$ is practically the same as that of the last term Δ_m. This shows that for large m, the waitingtime between the last two successes completely overshadows even the sum of all the previous waitingtimes.

M. N. Tata [7] has investigated the sequence L_m, $m = 1, 2, \cdots$ further and has shown, in particular, that the limiting distribution of $(L_{m+1})/(L_m)$ exists for $m \to \infty$, but even it has an infinite expected value. This shows that the penalty paid for making the balls indistinguishable is in the waitingtimes involved.

To end this discussion in the world of fairy tales, where it started, we may say that the Genie II will exhibit the kth ball, less than k drawings after it was placed in the urn, provided he knows the numbering on the balls. If he has to go by chance, he can still be certain to draw out any given ball eventually, but the number of drawings involved in each case will be large with considerable probability. Since the genies were doomed to this activity for an infinite length of time, anyway, it probably does not matter to them whether they are guided by knowledge or by chance!

Acknowledgement. The "paradox" of the genies was mentioned several years ago to the author by Professor Samuel Kaplan. He should certainly be thanked for this stimulating conversation piece.

Constructive comments by Professor Louis J. Cote are also gratefully acknowledged.

This paper contains an expanded version of a talk given by the author under the "Program of Visiting Lecturers in Statistics—1967–68" under the sponsorship of the American Statistical Association, Biometric Society and the Institute of Mathematical Statistics.

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WEAK SUFFICIENT CONDITIONS FOR FATOU'S LEMMA AND LEBESGUE'S DOMINATED CONVERGENCE THEOREM

H. R. VAN DER VAART, North Carolina State University and ELIZABETH H. YEN, Columbia University

0. Introduction. In many expositions the Lebesgue-Stieltjes integral, $\int f(x)\mu(dx)=\int f d\mu$, or briefly $\int f$, of a measurable function f is defined as the limit of a sequence of integrals $\int s_n d\mu$, where the s_n are simple functions which in some sense tend to f as $n\to\infty$. So, when we are interested in the limit of a sequence $\int f_n d\mu$ where all f_n are measurable (rather than simple) functions, we have to deal with a double limit process. The *monotone convergence theorem* (MCT), *Fatou's Lemma*, and Lebesgue's *Dominated Convergence Theorem* (DCT) belong in this category. In the literature these results are discussed under a variety of mostly too restrictive conditions (cf. Section 2 below), which we have found tend to obscure their true nature in the mind of many students. The aim of this note is to present Fatou's Lemma as a special case of the MCT, and the DCT as a special case of Fatou's Lemma, being as general as possible as to conditions of boundedness and finiteness and also to indicate a method by which to construct the dominating function in the DCT. Of these objectives the last one seems to have some novelty. However, our main concern is pedagogical.

1. Notations and terminology. All functions discussed are assumed to be defined on a totally σ-finite measure space $(X, \mathfrak{A}, \mu)$ into the extended real number system R^*. (For the properties of R^* see for instance [4], p. 2). All functions discussed will be *measurable* (i.e., if B is $\{+\infty\}$, $\{-\infty\}$, or a Borel subset of the real line R, then $f^{-1}(B)\in\mathfrak{A}$). Given ϕ: $X\to R^*$, the symbols ϕ^+ and ϕ^- have the usual meaning: $\phi^+=\frac{1}{2}(\phi+|\phi|)$, $\phi^-=\frac{1}{2}(-\phi+|\phi|)$, so that $\phi=\phi^+-\phi^-$. Integration is always over some set A belonging to the σ-algebra $\mathfrak{A}$. For our purposes the choice of A is irrelevant (all properties stated concerning integrands are to hold on A), and we shall omit all reference to it. Whenever we write $\int\phi$, or $\int\phi d\mu$, or $\int\phi(x)\mu(dx)$, we imply that the *integral* is *defined*, either as a finite number, or as $+\infty$, or as $-\infty$, and we shall call such a ϕ integrable. In fact, we shall say

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(a) $\int\phi^+$ finite, $\int\phi^-$ finite, $\int\phi=\int\phi^+-\int\phi^-$, ϕ *summable*,

(b) $\int\phi^+$ finite, $\int\phi^-$ infinite, $\int\phi=-\infty$,

(c) $\int\phi^+$ infinite, $\int\phi^-$ finite, $\int\phi=+\infty$.

A function ϕ such that $\lim_{n\to\infty}\phi_n(x)=\phi(x)$ for μ-almost all $x(\epsilon A)$ will be denoted by $\lim_n \phi_n$.

Given a sequence $\{\alpha_k\}$, define $\beta_n=\inf_{k\geqq n}\alpha_k$ and $\gamma_n=\sup_{k\geqq n}\alpha_k$. Then $\{\beta_n\}$ is a nondecreasing sequence and $\{\gamma_n\}$ is a nonincreasing sequence. So $\lim_{n\to\infty}\beta_n=\lim_n\beta_n$ and $\lim_{n\to\infty}\gamma_n=\lim_n\gamma_n$ exist (the first being finite or $+\infty$, the second being finite or $-\infty$). We shall write $\lim\inf_n \alpha_n$ for $\lim_n\beta_n$ and $\lim\sup_n\alpha_n$ for $\lim_n\gamma_n$. A function ϕ such that $\lim\inf_n\phi_n(x)=\phi(x)$ for μ-almost all x will be denoted by $\lim\inf_n\phi_n$; a similar definition applies to $\lim\sup_n\phi_n$. $\mathrm{Lim}_n\,\phi_n$ exists iff $\lim\inf_n\phi_n=\lim\sup_n\phi_n$ (μ-almost everywhere).

All equalities and inequalities between functions are supposed to hold μ-almost everywhere.

When citing other authors we shall translate their statements in terms of the above terminology and notation.

In all our examples and counterexamples, μ is understood to be Lebesgue measure.

2. Comparative survey of versions of Fatou's Lemma. Table 1 lists conditions and conclusions of a number of versions of Fatou's Lemma. Measurability of f_n is common to all assumptions and the inequality

$$\int \lim\inf_n f_n \leqq \lim\inf_n \int f_n \tag{1}$$

is common to all conclusions: these have been omitted from our list.

Analyzing this survey we make the following remarks:

(i) The conclusion $\int\lim_n f_n\leqq K$ is weaker than the usual one, but the proofs given actually prove inequality (1), as is pointed out by Riesz and Sz. Nagy.

(ii) Jeffery's version may suggest the question as to whether or not summability of $\lim_n f_n$ (or of $\lim\inf_n f_n$, as the case may be) could imply that $\lim\inf_n\int f_n<+\infty$. Actually Munroe claims that this is, indeed, true. (See the "only if" part of his Section 34.1.) A counterexample is provided by the sequence $\{f_n\}$:

$$f_n(x) = \begin{cases} n^2(n+1) & \text{for } (n+1)^{-1} < x < n^{-1} \\ 0 & \text{otherwise.}\end{cases}$$

Here $\lim\inf_n f_n(x)=\lim_n f_n(x)=0$ for all x, so $\lim\inf_n f_n$ is summable. Yet $\int_0^\infty f_n=n$; hence $\lim\inf_n\int f_n=\lim_n\int f_n=+\infty$.

(iii) Comparison of the Goffman-Pedrick version with those preceding it suggests the question as to whether it is possible that $f_n\geqq 0$ and f_n be summable for all n, yet $\lim\inf_n f_n$ be not summable. The following example shows the answer is in the affirmative:

$$f_n(x) = \begin{cases} x^{-2} & \text{for } 2^{-n} < x < 1 \\ 0 & \text{otherwise.}\end{cases}$$

TABLE 1

Kolmogorov-Fomin, § 44, Th. 2; Šilov-Gurevič, p. 37; Riesz-Sz.Nagy, § 20	$f_n \geqq 0$	$\int f_n \leqq K$	$\lim_n f_n$ exists		—	$\Rightarrow$ $\lim_n f_n$ summable and $\int \lim_n f_n \leqq K$
Jeffery, Theorem 4.17	$f_n \geqq 0$	$\int f_n < +\infty$	$\lim_n f_n$ exists: summable; not summable		—	$\Rightarrow$ (summable); $\Rightarrow \lim_n \int f_n = +\infty$ (not summable)
Berberian, Sect. 32; Halmos, Sect. 27, Th. F; Munroe, Sect. 34.1	$f_n \geqq 0$	$\int f_n < +\infty$		—	$\lim\ \inf_n \int f_n < +\infty$	$\Rightarrow \lim\ \inf_n f_n$ summable
McShane-Botts, Ch. 5, § 4, Ex. 3	$f_n \geqq g$ and g summable	f_n summable		—	$\lim\ \inf_n \int f_n < +\infty$	$\Rightarrow \lim\ \inf_n f_n$ summable
Loève, Sect. 7.2, Th. B	$h \geqq f_n \geqq g$ and g, h summable	hence f_n summable		—	—	$\Rightarrow$
Goffman-Pedrick, Sect. 3.7, Th. 1	$f_n \geqq 0$	$\int f_n < +\infty$		—	—	$\Rightarrow$
Royden, Ch. 11, Sect. 3, Th. 12	$f_n \geqq 0$	—	$\lim_n f_n$ exists		—	$\Rightarrow$
Dunford-Schwartz, Sect. III. 6.19; Hewitt-Stromberg, Ch. 3, (12-23); Rudin, Sect. 10.31; Saks, Ch. 1, Th. 12.10	$f_n \geqq 0$	—		—	—	$\Rightarrow$

Here $\int_0^1 f_n < +\infty$ for all n; yet $\liminf_n f_n(x) = x^{-2}$ for $0<x<1$. Hence $\liminf_n f_n$ is not summable.

(iv) Whenever the existence of $\lim_n f_n$ is assumed one may, as a matter of course, replace $\liminf_n f_n$ in inequality (1) by $\lim_n f_n$.

(v) It is clear that the version as given by Dunford and Schwartz and others (not requiring the f_n to be summable) can easily generate all versions providing for finiteness or even boundedness of certain integrals.

(vi) The inequality

$$\limsup_n \int f_n \leqq \int \limsup_n f_n \tag{2}$$

constitutes a complement to inequality (1). The substitution $f_n = -f_n^*$ shows that (2) holds true under the condition $f_n \leqq 0$ (also see Corollary to Theorem 1 in Section 3.7 of Goffman and Pedrick). Thus inequalities (1) and (2) would hold for two mutually exclusive classes of functions. This, fortunately, is not true: the condition $f_n \geqq 0$ for inequality (1) is unduly restrictive, as can be seen from the conditions cited by McShane and Botts, and by Loève. In fact, as pointed out by Loève, the condition, "$h \geqq f_n \geqq g$, and g and h summable," cited in Table 1, is sufficient for both (1) and (2). Note that $h \geqq f_n$ is irrelevant for the validity of (1), and $f_n \geqq g$ is irrelevant for the validity of (2). Applications of these alternative conditions are, however, hampered by the fact that there is no construction for deriving g and h from the sequence $\{f_n\}$. Our discussion will point up a natural construction for precisely this purpose.

3. Monotone convergence theorem. *Let $\{\phi_n\}$ be a nondecreasing sequence of extended real valued nonnegative measurable functions defined on the measure space $(X, \mathfrak{A}, \mu)$. Then*

$$\int \lim_n \phi_n = \lim_n \int \phi_n. \qquad \text{(MCT)} \tag{3}$$

Proof. The proof depends on the way the integral is defined. In the present context the simplest definition is the one used by Loève (Section 7.1) and by Goffman and Pedrick (Section 3.6). Their proofs of MCT (see Loève, Section 7.2, Theorem A, and Goffman and Pedrick, Section 3.7, Proposition 1) are closely related and exhibit the double limit process mentioned in our section 0 in an explicit way.

Our first concern is to weaken the condition $\phi_n \geqq 0$. We cannot just omit this condition as is clear from the following example of an increasing sequence of functions ϕ_n:

$$\phi_n(x) = -1/nx \quad \text{for } 0 < x < 1. \tag{4}$$

Here $\int_0^1 \phi_n = -\infty$ for all n, so $\lim_n \int_0^1 \phi_n = -\infty$, whereas $\lim_n \phi_n(x) = 0$ for $0<x<1$, so $\int_0^1 \lim_n \phi_n = 0$. The following generalization of MCT is easily proved:

MCT^g. *Let $\{\phi_n\}$ be a nondecreasing sequence of extended real valued measurable functions defined on $(X, \mathfrak{A}, \mu)$. If at least one natural number k exists for which* *either* $\int\phi_k=+\infty$, *or* ϕ_k *is summable, it follows that*

$$\int \lim_n\phi_n = \lim_n \int \phi_n. \tag{5a}$$

If neither of these conditions is satisfied we can only conclude that

$$\int \lim_n\phi_n \geqq \lim_n \int \phi_n. \tag{5b}$$

Proof. If a natural number k exists for which $\int\phi_k=+\infty$, then both members of (5a) are seen to be equal to $+\infty$, since $\lim_n \phi_n \geqq \phi_k$ and since for $n\geqq k$ we have $\int\phi_n\geqq\int\phi_k$. If a natural number k exists for which ϕ_k is summable, the sequence $\{\phi_n-\phi_k\}_{n\geqq k}$ satisfies the conditions for the MCT; hence

$$\lim_n \int (\phi_n - \phi_k) = \int (\lim_n\phi_{\,n\to\phi} - \phi_k).$$

Addition of the *finite number* $\int\phi_k$ to both sides of this equality completes this part of the proof. Finally, if neither condition holds, then $\int\phi_n=-\infty$ for all n, so that (5b) is trivially true. The example defined by equation (4) shows that in this case the inequality sign may hold in (5b), whereas examples of strict equality are trivial.

The reader can easily supply a result *symmetric to* MCT^g by replacing *nondecreasing* by *nonincreasing*, $+\infty$ by $-\infty$, and $\geqq$ by $\leqq$. We will refer to the result discussed in the second half of this section as $\uparrow$ MCT^g, and to its symmetric counterpart as $\downarrow$ MCT^g.

The reader will observe that MCT follows from $\uparrow$ MCT^g by adjoining a member ϕ_0 to the sequence, where $\phi_0(x)=0$ for all x, and putting the number k equal to 0. Of course this is possible only if $\phi_n\geqq 0$ for all $n=1, 2, 3, \cdots$.

4. Fatou's Lemma and the dominated convergence theorem. The following results refer to arbitrary sequences $\{f_n\}$. They follow immediately by applying $\uparrow$ MCT^g to the nondecreasing sequence $\{g_n\}=\{\inf_{\nu\geqq n} f_\nu\}$ and $\downarrow$ MCT^g to the nonincreasing sequence $\{h_n\}=\{\sup_{\nu\geqq n} f_\nu\}$.

FATOU I. *Let* $\{f_n\}$ *be a sequence of extended real valued measurable functions defined on* $(X, \mathfrak{A}, \mu)$. *If at least one natural number* k *exists for which either* $\int g_k=\int\inf_{\nu\geqq k} f_\nu=+\infty$ *or* $g_k=\inf_{\nu\geqq k} f_\nu$ *is summable, then*

$$\int \liminf_n f_n \leqq \liminf_n \int f_n. \tag{6}$$

Proof. Applying $\uparrow$ MCT^g to $\{g_n\}$ we find

$$\int \liminf_n f_n = \lim_n \int g_n. \tag{7}$$

Now

$$g_n(x) = \inf_{\nu\geqq n} f_\nu(x) \leqq f_m(x) \text{ for all } m \geqq n.$$

So

$$\int g_n \leqq \int f_m \quad \text{for all } m \geqq n;$$

hence

$$\int g_n \leqq \inf_{m\geqq n} \int f_m.$$

Consequently

$$\lim_n \int g_n \leqq \lim\inf_n \int f_n. \tag{8}$$

Combination of (7) and (8) yields inequality (6) of Fatou I.

If the conditions of Fatou I do not hold, $\uparrow$ MCT^g shows that (7) should be replaced by

$$\int \lim\inf_n f_n \geqq \lim_n \int g_n. \tag{7*}$$

Inequality (8) remains valid. Combination of (7*) and (8) now seems to imply that a sequence $\{f_n\}$ with

$$\int g_k = \int \inf_{\nu\geqq k} f_\nu = -\infty \quad \text{for all natural } k$$

may satisfy inequality (6) (with $\leqq$) as well as its opposite (with $\geqq$). In fact, in Section 5 we will give an example of both types of sequences. Hence a *necessary* condition for the validity of Fatou's inequality (6) cannot be given in terms of integrability and/or summability of the members of the sequence $\{g_n\} = \{\inf_{\nu\geqq n} f_\nu\}$ alone. However, the existence of a number k with the property cited in our Fatou I constitutes the sharpest *sufficient* condition to date.

The reader will have no difficulty in proving

FATOU II. *Let $\{f_n\}$ be a sequence of extended real valued measurable functions defined on $(X, \mathfrak{A}, \mu)$. If at least one natural number k exists for which either $\int h_k = \int \sup_{\nu\geqq k} f_\nu = -\infty$ or $h_k = \sup_{\nu\geqq k} f_\nu$ is summable, then*

$$\lim\sup_n \int f_n \leqq \int \lim\sup_n f_n. \tag{9}$$

Indeed he need only apply $\downarrow$ MCT^g to the sequence $\{h_n\}$.

Evidently, the best *sufficient* condition for the validity of both (6) and (9) is given by

FATOU III. *Let $\{f_n\}$ be a sequence of extended real valued measurable functions defined on $(X, \mathfrak{A}, \mu)$. If at least one natural number k exists for which both $g_k = \inf_{\nu\geqq k} f_\nu$ and $h_k = \sup_{\nu\geqq k} f_\nu$ are summable, then*

$$\int \liminf_n f_n \leqq \liminf_n \int f_n \leqq \limsup_n \int f_n \leqq \int \limsup_n f_n. \tag{10}$$

Indeed, in order that Fatou II may apply, $\int h_k = \int \sup_{\nu \geqq k} f_\nu$ has to come down from $+\infty$ for some k, and in order that Fatou I may apply, $\int g_k = \int \inf_{\nu \geqq k} f_\nu$ has to come up from $-\infty$ for some k. Hence $\int h_k$ cannot ever get down to $-\infty$, and $\int g_k$ cannot ever get up to $+\infty$.

If to the conditions of Fatou III is added the condition that $\lim_n f_n$ exist, the string of inequalities in (10) becomes a string of equalities. This leads to our version of the dominated convergence theorem:

DCT. *Let* $\{f_n\}$ *be a sequence of extended real valued measurable functions defined on* $(X, \mathfrak{A}, \mu)$. *If* $\lim_n f_n$ *exists* (μ-*almost everywhere*) *and if, in addition, at least one natural number* k *exists for which both* $g_k = \inf_{\nu \geqq k} f_\nu$ *and* $h_k = \sup_{\nu \geqq k} f_\nu$ *are summable, then*

$$\lim_n \int f_n \textit{ exists, and} \tag{11}$$

$$\lim_n \int f_n = \int \lim_n f_n \textit{ is finite.} \tag{12}$$

By hindsight we can see that the functions g_k and h_k are as closely dominating functions as one can hope to obtain, so in critical problems this version offers more hope for success than the classical requirement that one find some summable function s such that $|f_n| \leqq |s|$. In fact, our version supplies a construction of a sequence from which one can choose a dominating function s.

One might wonder, since g_k and h_k have to be summable for *some* k (no matter how large), if not summability of $\liminf_n f_n$ and of $\limsup f_n$ would be sufficient. Counterexample (Hartman and Mikusiński, Chapter iv, Section 3):

$$f_n(x) = \begin{cases} -2xn^2 + 2n & \text{for } 0 < x \leqq n^{-1} \\ 0 & \text{for } n^{-1} < x \leqq 1. \end{cases}$$

Here $\lim_n f_n(x) = 0$ for all $x \in]0, 1]$; $\int_0^1 \lim_n f_n(x)dx = 0$; $\int_0^1 f_n(x)dx = 1$ for all n; $\lim_n \int_0^1 f_n(x)dx = 1$. So $\lim\sup_n f_n = \lim\inf_n f_n$ is summable, yet (12) does not apply. What is wrong? The function $\sup_{\nu \geqq k} f_\nu$ is not summable for any k as can be shown by elementary computation.

5. Two sequences of functions violating the conditions of Fatou I. The increasing sequence of functions defined in equation (4) has the following properties:

$$g_k(x) = \inf_{\nu \geqq k} \phi_\nu(x) = -(kx)^{-1},$$

$$\int_0^1 g_k = -\infty \quad \text{for all natural } k,$$

$$\int_0^1 \lim\,\inf_n \phi_n = \int_0^1 \lim_n \phi_n = 0,$$

$$\lim\,\inf_n \int_0^1 \phi_n = \lim_n \int_0^1 \phi_n = -\infty.$$

Hence $\int \lim\,\inf_n \phi_n > \lim\,\inf_n \int \phi_n$, which constitutes an example of a sequence of functions violating the condition of Fatou I, for which the Fatou inequality (6) is invalid.

Next we will construct a sequence $\{f_n\}$ violating the condition of Fatou I, for which, however, the Fatou inequality (6) is still valid. Let $\{r_n\}$ be the sequence of all rational numbers between 0 and 1, constructed in the following manner: write all proper fractions, reduced to lowest terms, first choosing those with denominator 2 (so $r_1 = 1/2$), then those with denominator 3 (so $r_2 = 1/3$, $r_3 = 2/3$), then those with denominator 4 (hence $r_4 = 1/4$, $r_5 = 3/4$), etc. Let $\rho \in]1, 2[$ be a fixed real number. Now define

$$f_n(x) = \begin{cases} -1/x & \text{if } r_n \cdot \rho^{-1} < x < r_n \cdot \rho, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

This sequence of functions has the following properties:

$$\inf_{\nu \geq k} f_\nu(x) = -1/x \quad \text{for all natural } k, \text{ all irrational } x \in]0, \ \rho[; \tag{14}$$

hence $\int_0^2 \inf_{\nu \geq k} f_\nu = -\infty$. Also $\lim\,\inf_n f_n(x) = -1/x$ for $0 < x < \rho$; so $\int_0^2 \lim\,\inf_n f_n = -\infty$. Finally $\int_0^2 f_n = -2 \log \rho$; so $\lim\,\inf_n \int_0^2 f_n = -2 \log \rho$. Clearly $\int \lim\,\inf_n f_n < \lim\,\inf_n \int f_n$, which we wanted to show.

Proof of equality (14). For any irrational number x there exists an *infinite* number of fractions p/q such that $x \in]p/q - 1/q^2, p/q + 1/q^2[$. (See Perron, Chapter 5, Section 36.) Since for $q > \rho/(\rho - 1)$ we have that $(p/q)(1/\rho) < p/q - 1/q^2$ and that $(p/q)\rho > p/q + 1/q^2$, it follows that each irrational number $x \in]0, 1[$ is covered by an infinite number of intervals of the form $](p/q)(1/\rho), (p/q)\rho[$, with $0 < p < q$, p and q relatively prime, and fixed $\rho \in]1, 2[$.

Since the members of the above-constructed sequence $\{r_n\}$ are irreducible fractions whose denominators form a nondecreasing sequence of natural numbers it follows that each irrational $x \in]0, 1[$ is covered by an infinite number of intervals of form $]r_n/\rho, r_n\rho[$. Note that the irrational $x \in]1, \rho[$ are trivially so covered: just take $r_n = (n-1)/n$ with n sufficiently large. Hence definition (13) shows that for any irrational $x \in]0, \rho[$, no matter how large N may be, there is always a number $n > N$ with $f_n(x) = -1/x$. This proves equality (14).

This work was partially supported by Public Health Service Grant GM-678 from the National Institute of General Medical Sciences to the Biomathematics Training Program, Institute of Statistics, Raleigh Section, and partially supported by the Biology Branch of the Office of Naval Research under Grant No. 2249(05).

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ON BASES AND CYCLES

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Introduction. The relations between an integer and the digits used to represent the integer in another base of numeration have always been of interest to mathematicians. Consider for instance 23 in the decimal system. The sum of the squares of the digits of this number is 13 which is different from 23. In fact, one may easily prove that there exists no integer which equals the sum of the squares of its digits in the decimal system. (See [3].) It is shown in [4] that if $b(>1)$ is such that b^2+1 is composite, then there exist integers which equal the sum of the square of their digits representing them in the base b.

In Section 1 of this paper we shall study some properties of such integers. In Section 2, we shall study certain properties of integers which equal the sum of the cubes of the digits representing them in a given base $b(>1)$. Such numbers exist in base ten. For instance $153=1^3+5^3+3^3$. Finally in Section 3 we shall generalize some of our results and introduce the reader to a very curious and interesting arithmetical function, which is the aim of this paper. In what follows, $b>1$.

Section 1. We begin with

DEFINITION 1.1. *Let a positive integer A have the representation* $(a_n, a_{n-1}, \cdots,$

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$a_0)_b$ *in the base b. We call A a digit square integer in the base b (and say the base b has the digit square integer A) if* $A=a_0^2+a_1^2+\cdots+a_n^2$.

It is easily proved that except for the trivial digit square integer 1, any other digit square integer has exactly two digits. It follows from Definition 1.1 that if $(x, y)_b$ is a digit square integer in the base b, then

$$x^2+y^2-bx-y=0. \tag{1.1}$$

By solving this equation as a quadratic in x and also in y one easily notes that the number $D(b)$ of digit square integers in the base b is even.

By completing squares in (1.1) and calling $2x-b=u$ and $2y-1=v$, we obtain

$$u^2+v^2=b^2+1. \tag{1.2}$$

Let b be odd. It follows easily that u and v are both odd. Further $v>1$, and b^2+1 has no odd divisors of the form $4j+3$. In general, in an equation of the form (1.2), the number of solutions or, what is the same thing, the number of representations of b^2+1 as the sum of two squares, is known to be

$$4\sum_t (-1)^{(t-1)/2}$$

where the summation is taken over all positive odd divisors of b^2+1. In our case, however, in view of the foregoing considerations this expression, which actually gives the number $D_0(b)$ of the number of digit square integers in the odd base b, may be written as

$$D_0(b)=2\left(\sum_t 1\right)-2.$$

Let us call a divisor t of b^2+1 proper if $1<t<b^2+1$. Since $t=1$ is an obvious odd divisor, we have

$$D_0(b)=2\sum_t 1,\ t \text{ a proper odd divisor of } b^2+1. \tag{1.3}$$

Similarly we find the number, $D_e(b)$, of digit square integers in an even base b to be

$$D_e(b)=\sum_t 1. \tag{1.4}$$

THEOREM 1.2. *In any given base b, the number D(b) of digit square integers is given by* $D(b)=\sum_t 1$, *the summation taken over all positive proper divisors t of* b^2+1.

Proof. The case when b is even is obvious from (1.4). If b is odd, $b^2+1=2p$ where p is odd. If $A=\{p, t_1, t_2, \cdots, t_n\}$ is the set of proper odd divisors of b^2+1, then $B=\{2, 2t_1, \cdots, 2t_n\}$ is seen to be the set of proper even divisors of b^2+1. A and B have the same number of elements and the theorem follows from (1.3).

COROLLARY. *A given base b can have digit square integers if and only if* b^2+1 *is not a prime.* (See [4].)

THEOREM 1.3. *Let b be odd. The sum, S, of all the digit square integers in the base b is equal to* b^2+b+1 *if and only if* $\frac{1}{2}(b^2+1)$ *is a prime.*

Proof. Let $b=2m+1, m>1$. The integers $k=(m, m+1)_b$ and $l=(m+1, m+1)_b$ are digit square integers as is easily verified. The theorem follows from the following statements which are equivalent:

$$\tfrac{1}{2}(b^2+1) \text{ is a prime, } D_0(b) = 2, S = k + l = b^2 + b + 1.$$

Section 2. Analogous to the concept of a digit square integer we have the digit cube integer.

DEFINITION 2.1. *An integer* $A=(a_n, a_{n-1}, \cdots, a_0)_b$ *is called a digit cube integer in the base b (and the base b is said to have the digit cube integer A) if* $A=a_0^3+a_1^3+\cdots+a_n^3$.

A digit square integer has precisely two digits. The corresponding theorem in the case of digit cube integers is

THEOREM 2.1. *In a given base b, a digit cube integer cannot have more than four digits. If the last digit is zero or one, it cannot have more than three digits and if* $\sqrt{b}$ *is irrational, it has exactly three digits.*

Proof. Let $A=(a_p, a_{p-1}, \cdots, a_0)_b$, $a_p\neq 0$, $0\leqq a_i\leqq b-1$, $p>1$ be a digit cube integer so that

$$\sum_{r=0}^{p} a_r(b^r - a_r^2) = 0. \tag{2.1}$$

In this summation each term increases with respect to a_r if $r\geqq 3$ and the least positive value of such a term is b^r-1. We write (2.1) as

$$\sum_{r=2}^{p} a_r(b^r - a_r^2) = a_1(a_1^2 - b) + a_0(a_0^2 - 1).$$

The terms on the right increase respectively with respect to a_1 and a_0 (verified by taking derivatives) and hence the maximum value v of the right side occurs when $a_1=a_0=b-1$. We have

$$v = 2b^3 - 7b^2 + 6b - 1 < b^4 - 1.$$

Hence $a_r\neq 0$ for some $r\geqq 4$ shows (2.1) is impossible and thus we must have $r<4$.

If $a_0=0$ or 1, $v=b^3-4b^2+4b-1$ and as above, for (2.1) to be possible in this case, we must have $r\leqq 2$. If A has only two digits, from (2.1) we have $b=a_1^2$; that is, b is a perfect square. Hence if $\sqrt{b}$ is irrational, $a_2\neq 0$ and $r=2$ which proves the theorem.

A general discussion of digit cube integers involves the solution in positive integers of certain types of cubic and biquadratic equations, usually very difficult to solve. Here we restrict our attention to the class of digit cube integers with last digit zero or one. If $\sqrt{b}$ is irrational, then by the theorem just proved, the digit cube integer has only three digits. If $\sqrt{b}$ is rational, $(\sqrt{b}, 0)_b$ and $(\sqrt{b}, 1)_b$ are the only digit cube integers with two digits and this case is not of much interest. We now prove

THEOREM 2.2. *Let p be a given positive integer and (r_1, y_1) the minimal positive solution of the Pell's equation*

$$x^2 - 4y^2p(1 + p^3) = 1.$$

Let $A = p(1+p^3)$ and define r_n and y_n by

$$r_n + 2y_n\sqrt{A} = (r_1 + 2y_1\sqrt{A})^n, \qquad n \geqq 1$$

If $b_n = (r_n - 1)/2p$ is a positive integer and $py_n < b_n$, then $(py_n, y_n, 0)_{b_n}$ and $(py_n, y_n, 1)_{b_n}$ are digit cube integers in the base b_n.

Proof. If $(x, y, 0)_b$, $(x, y, 1)$, $y \neq 0$ are digit cube integers in the base b, we must have

$$x(b^2 - x^2) + y(b - y^2) = 0.$$

In the special case when $x = py$, this becomes

$$y^2(1 + p^3) - b^2p - b = 0.$$

Write $2bp + 1 = r$, a positive integer, and we have

$$r = 2bp + 1 = \sqrt{1 + 4y^2p(1 + p^3)}$$

or

$$r^2 - 4Ay^2 = 1, \qquad A = p(1 + p^3). \tag{2.4}$$

Clearly, A is not a perfect square, so that (2.4) is Pell's equation. The theorem now follows since every solution (r_n, y_n) of (2.4) may be expressed in terms of the minimal positive solution (r_1, y_1), $y_1 \neq 0$ and r_n and b_n are related by $r_n = 2b_np + 1$.

The transformation $y = px$ in the equation $x^3 + y^3 - b^2x - by = 0$ leads one to the more complicated form of Pell's equation, $x^2 - Ay^2 = N$ which may or may not have a solution. In any case we do not obtain as striking a result as Theorem 2.2.

Section 3. Although the decimal system has no digit square integers, Arthur Porgess [1] discovered an interesting property, viz., that if we take any positive integer and take the sum of the squares of its digits, do the same thing to the latter number and keep iterating the process, sooner or later we will get one of the following numbers:

$$\{1, 20, 4, 16, 37, 58, 89, 145, 42\}.$$

This is a very interesting set and is in fact a cycle in a sense. That is except for 1, if we take any member A of the set and apply the process described above we will get each of the integers following A in the above set and at the eighth step get back A.

If we consider cubes of the digits, $\{55, 250, 133\}$ is a cycle. In base 7, $\{(2)_7, (4)_7, (2, 2)_7, (1, 1)_7\}$ is a cycle and in base 8 $\{(4)_8, (2, 0)_8\}$ is a cycle when we consider the sum of the squares of the digits.

One may wonder if there are cycles in every base when one considers the sum of the mth powers of digits. The answer is in the affirmative as will be proved in this section. We need a few definitions.

DEFINITION 3.1. *Let m be a positive integer and $A=(a_p, a_{p-1}, \cdots, a_0)_b$ any integer in a given base of numeration b, $a_p\neq 0$, $p\geqq 1$, $0\leqq a_j\leqq b-1$, $0\leqq j\leqq p$. The numerical function $S(b, m, A)$, defined for every positive integer in the base b, is defined as*

$$S(b, m, A) = \sum_{r=0}^{p} a_r^m \tag{3.1}$$

and recursively by

$$S^n(b, m, A) = S(b, m, S^{n-1}(b, m, A)) \tag{3.2}$$

for every positive integer $n\geqq 2$.

In other words, $S(b, m, A)$ operating on an integer A takes the sum of the mth powers of its digits.

DEFINITION 3.2. *The number of digits of a given integer A in a base b is denoted by* dig A.

THEOREM 3.1. *Let m be a given positive integer and A any positive integer in the base b. If $A\geqq b^{m+1}$, then*

$$S(b, m, A) < A.$$

Proof. Let $A=(a_n, a_{n-1}, \cdots, a_0)_b$ so that dig $A=n+1$, $0\leqq a_j\leqq b-1$ and $a_n\neq 0$. We have

$$A = \sum_{r=0}^{n} a_r b^r$$

so that

$$D = A - S(b, m, A) = \sum_{r=0}^{n} a_r(b^r - a_r^{m-1}). \tag{3.3}$$

Now for a term in (3.3) to be negative we must have $b^r<a_r^{m-1}$, that is $(b^r/a_r^{m-1})<1$. Also $b^{r-m+1}=(b^r/b^{m-1})<(b^r/a_r^{m-1})$. This means $r-m+1<0$; hence $r\leqq m-2$.

We may assume that $b^r<a_r^{m-1}$ for all $r\leqq m-2$ and write (3.3) as

$$D = \sum_{r=m-1}^{n} a_r(b^r - a_r^{m-1}) - \sum_{r=0}^{m-2} a_r(a_r^{m-1} - b^r). \tag{3.4}$$

The second sum can have a possible maximum value V when $a_r = b-1$ for all r. It is easy to verify that

$$V = (m - 1)(b - 1)^m - b^{m-1} + 1.$$

We have $A \geqq b^{m+1}$ so that dig $A \geqq 2m+2$; that is, $n \geqq m+1$. Clearly $a_n(b^n - a_n^{m-1})$ increases with respect to a_n and thus its least positive value is $b^n - 1$. If $m = 1$, the theorem is trivial so let $m \geqq 2$. We observe that

$$b^n - 1 - V = b^n - 1 - (m - 1)(b - 1)^m - 1 + b^{m-1} > 0.$$

It follows from (3.4) that $D > 0$ and this proves the theorem.

Let $A' = ((b-1), (b-1), \cdots, (b-1))_b$ with dig $A' = r$. Then if A is any integer with dig $A = r$, it follows that $S(b, m, A) \leqq S(b, m, A') = r(b-1)^m$, whence, by taking logarithms to base b, we obtain

$$\text{dig } S(b, m, A) \leqq 1 + \log_b r + m \log_b(b - 1).$$

This gives the size of $S(b, m, A)$ relative to A. In particular, if $b = 10$, $m = 2$, dig $A = 10$, then $S(b, m, A)$ has less than four digits.

We shall now introduce the concept of a *cycle.*

DEFINITION 3.3. *Let m be a given positive integer and b a given base of numeration. Let C be a set of positive integers in the base b. We call C a cycle if for every A belonging to C there exists a positive integer $p(A)$ such that $A = S^p(b, m, A)$.*

DEFINITION 3.4. *The length of a cycle C in a base b is the number of elements in C.*

DEFINITION 3.5. *Let A be a given positive integer in the base b. The sequence $\{A_n\}$ defined by $A = A_0$, $A_n = S^n(b, m, A)$ is called the fundamental sequence of A.*

THEOREM 3.2. *Let A be a given positive integer in the base b. There exists a nonnegative integer $q(A)$ such that A_q is an element of the fundamental sequence of A and $A_q < b^{m+1}$.*

Proof. Obvious from Theorem 3.1.

DEFINITION 3.6. *Let A be a given positive integer in the base b. If A_{p_1}, A_{p_2} belong to the fundamental sequence of A, where $0 \leqq p_1 < p_2$ and such that $A_{p_1} = A_{p_2}$, then A is said to generate the cycle $\{A_{p_1}, A_{p_1+1}, \cdots, A_{p_2-1}\}$.*

THEOREM 3.3. *Every positive integer in a base b generates a unique cycle and there are only a finite number of cycles in that base.*

Proof. Consider the case when $A < b^{m+1}$. Let M be the finite set of all positive integers in the base b that are less than b^{m+1}. Let $\{A_n\}$ be the fundamental sequence of A and $K = \{A_{K_1}, \cdots, A_{K_s}\}$ its finite subsequence such that the elements of K are distinct elements of M. Consider A_{K_s+1}. By Theorem 3.2, there is an integer $p \geqq 0$ with $A_{K_s+1+p} < b^{m+1}$. The set K exhausts all such integers so that there exists a j such that $A_{K_j} \in K$ and $A_{K_j} = A_{K_s+1+p}$. Clearly A generates the cycle $\{A_{K_j}, \cdots, A_{K_s+p}\}$. The uniqueness is obvious.

For any general A, there is a least nonnegative integer l such that $A_l < b^{m+1}$. From what has been proved above there exist integers p_1, p_2, nonnegative, where $p_1 < p_2$ and $A_{l+p_1} = A_{l+p_2}$. It follows that A and A_l generate the same cycle and we are done.

It follows from the above proof that to obtain all cycles in a base b, we need consider the cycles generated by the elements of the finite set M. Hence there can be only a finite number of cycles and this completes the proof.

Conclusion. The operator $S(b, m, A)$ is an interesting arithmetical function about which we really don't know very much. It will be interesting to know if we can find out the number of cycles for given m and b and also the lengths of such cycles. A weaker question would be to find if for a given n there exists a cycle of length n. The answers appear to be very difficult to obtain.

In this paper, cycles of length one, viz., digit square integers ($m=2$), have been fully considered. A few results have been given in the case of digit cube integers ($m=3$).

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2. Problem E 1810, Amer. Math. Monthly, 72 (1965) 781.
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COMBINATIONS AND SUCCESSIONS

C. A. CHURCH, JR., University of North Carolina at Greensboro

In an earlier paper in this magazine [**1**] M. Abramson and W. Moser gave two results for restricted combinations, (4) and (5) below. In [**2**] they remark that the proof of (5) is incorrect. We derive these results by a simple extension of the first problem in Riordan [**4**, p. 14] and obtain some generalizations.

For notation, terminology, and basic combinatorial results we follow Riordan [**4**].

We use the

LEMMA [**4**, p. 92]. *The number of ways of distributing n like objects into m different cells is*

$$\binom{n+m-1}{m-1}. \tag{1}$$

In his quick solution to the "problème des ménages" Kaplansky [**3**], see also [**4**, p. 198], gives two results for restricted combinations. We state them in the

For any general A, there is a least nonnegative integer l such that $A_l < b^{m+1}$. From what has been proved above there exist integers p_1, p_2, nonnegative, where $p_1 < p_2$ and $A_{l+p_1} = A_{l+p_2}$. It follows that A and A_l generate the same cycle and we are done.

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Conclusion. The operator $S(b, m, A)$ is an interesting arithmetical function about which we really don't know very much. It will be interesting to know if we can find out the number of cycles for given m and b and also the lengths of such cycles. A weaker question would be to find if for a given n there exists a cycle of length n. The answers appear to be very difficult to obtain.

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THEOREM. *The number of k-combinations of the first n natural numbers, on a line, with no two consecutive integers in the same combination is*

$$\binom{n-k+1}{k}; \tag{2}$$

if arranged on a circle, so that n and 1 are also consecutive, the number is

$$\frac{n}{n-k}\binom{n-k}{k}. \tag{3}$$

For example, with $n=7$ and $k=3$ the 10 combinations of (2) are

135 136 137 146 147

157 246 247 257 357.

Delete 137, 147, and 157 from this list to get the 7 circular combinations of (3).

In the combinations above the integers occur separately, i.e., in k blocks, each of length one. Following Abramson and Moser [1] we shall consider the case of r blocks, each a sequence of consecutive integers. For example, 124569 has the three blocks 12, 456, and 9 of length 2, 3, and 1, respectively. We state the results of Abramson and Moser in the following:

THEOREM. *The number of k-combinations of the first n natural numbers, on a line, with exactly r blocks of consecutive integers is*

$$\binom{k-1}{r-1}\binom{n-k+1}{r}; \tag{4}$$

if arranged on a circle, so that n and 1 are consecutive, the number is

$$\frac{n}{n-k}\binom{k-1}{r-1}\binom{n-k}{r}. \tag{5}$$

For example, with $n=7$, $k=3$, and $r=2$ the 20 combinations of (4) are

134	245	356	467	124	235	346	457
145	256	367		125	236	347	
156	267			126	237		
167				127			

Delete 167 and 127 and add 137, 147, and 157 to get the 21 circular combinations of (5). Here 7 and 1 form a block of length two.

In proving (5) we get, along the way, the

THEOREM. *The number of circular k-combinations of the first n natural numbers with exactly r blocks of consecutive integers in which n and 1 never occur together is*

$$\frac{n-k+r}{n-k}\binom{k-1}{r-1}\binom{n-k}{r}. \tag{6}$$

In the example above the 18 combinations of (6) are obtained by deleting 167 and 127 from the list.

Comparison of (5) and (6) yields the

COROLLARY. *The number of circular k-combinations of the first n natural numbers with exactly r blocks of consecutive integers in which n and* 1 *always occur in a block is*

$$\frac{k-r}{n-k}\binom{k-1}{r-1}\binom{n-k}{r}. \tag{7}$$

In the example these are the combinations 137, 147, and 157.

For contrast between (5) and (6) take $n=7$, $k=3$, and $r=1$. The 5 combinations of (6), which agree with (4) when $r=1$ and $k<n$, are

$$123,\ 234,\ 345,\ 456,\ 567;$$

whereas (5) gives in addition 167 and 127. It is perhaps clearer to indicate these two as 671 and 712.

Note that (5) and (6) both agree with (3) when $r=k$. This follows from the fact that the contribution from (7) is zero.

We base our derivations on (1) and arrangements of p plus signs and q minus signs on a line (dashes and dots in [1]). If $p=k$ and $q=n-k$, each arrangement of p pluses and q minuses on a line corresponds in a one-one way with a k-combination of the first n natural numbers as follows. Arrange the first n natural numbers on a line in their natural (rising) order; place a plus sign under each integer selected and a minus sign under each integer not selected.

To fix the idea we rederive (2) and (3).

Riordan [4, p. 14] poses the following problem, with pluses and minuses interchanged. In how many ways can p pluses and q minuses, $q \geqq p-1$, be placed on a line with no two pluses together? Place the p pluses and $p-1$ of the minuses on a line with a minus sign between each pair of pluses. Distribute the remaining $q-(p-1)$ minuses in the $p+1$ cells formed by the p pluses ($p-1$ between the pluses, one before the first, and one after the last). By (1) this can be done in

$$\binom{q+1}{p} \tag{8}$$

ways. With $p=k$ and $q=n-k$ this is (2).

To get (3) we want the number of arrangements of p pluses and q minuses on a line with no two consecutive pluses and such that the first and last signs are not both pluses. There are two mutually exclusive cases—an initial plus or an initial minus. In the first case the initial plus must be followed by a minus and the terminal sign must be a minus.

This is precisely (8) with p replaced by $p-1$ and q replaced by $q-2$, i.e.,

$$\binom{q-1}{p-1} = \frac{p}{q}\binom{q}{p}.$$

The second case is simply (8) with q replaced by $q-1$, i.e.,

$$\binom{q}{p}.$$

Thus there is a total of

$$\frac{p+q}{q}\binom{q}{p} \tag{9}$$

such arrangements. Put $p=k$ and $q=n-k$ in (9) to get (3).

We are able to get the circular combinations from those on a line by considering them as being wrapped around a circle. Then n and 1 are consecutive if they are in the same combination.

It is now easy to get (4). In each of the $\binom{q+1}{r}$ arrangements of r pluses and q minuses on a line with no two pluses together of (8) the r pluses determine r cells. Into these r cells distribute the remaining $p-r$ pluses in

$$\binom{p-r+r-1}{r-1}=\binom{p-1}{r-1}$$

ways. Thus there are

$$\binom{p-1}{r-1}\binom{q+1}{r} \tag{10}$$

arrangements of p pluses and q minuses on a line with exactly r blocks of consecutive pluses. Put $p=k$ and $q=n-k$ in (10) to get (4).

We next obtain (6). Exactly as in the derivations of (10) take the

$$\frac{q+r}{q}\binom{q}{r}$$

circular arrangements of (9) and distribute the remaining $p-r$ pluses in the r cells to get

$$\frac{q+r}{q}\binom{p-1}{r-1}\binom{q}{r}. \tag{11}$$

Put $p=k$ and $q=n-k$ to get (6).

To get (5) we must add to (11) all those arrangements which begin and end with a plus. In terms of combinations n and 1 can appear together, but when they do they are to be considered as part of a block of length at least two. Thus we want the number of arrangements of $r+1$ pluses and q minuses on a line with no two pluses together and with initial and terminal pluses. The initial plus must be followed by a minus, and the terminal plus must be preceded by a minus. By (8), with q replaced by $q-2$ and $p=(r+1)-2=r-1$ we have

$$\binom{q-1}{r-1}=\frac{r}{q}\binom{q}{r}$$

such arrangements. Again distribute the remaining $p-(r+1)$ pluses in the $r+1$ cells to get the factor

$$\binom{p-1}{r} = \frac{p-r}{r}\binom{p-1}{r-1}.$$

Thus there is a total of

$$\frac{p-r}{q}\binom{p-1}{r-1}\binom{q}{r} \tag{12}$$

such arrangements.

Add (11) and (12) to get

$$\frac{q+p}{q}\binom{p-1}{r-1}\binom{q}{r}. \tag{13}$$

With $p=k$ and $q=n-k$ this is (5).

With very slight alterations in the derivations we can generalize all the preceding results in the following way.

First, restate Riordan's problem. In how many ways can p pluses and q minuses be placed on a line with at least b minuses between any two pluses? Here $b \geqq 0$ and $q \geqq b(p-1)$. Proceeding exactly as in the derivation of (8) we get the solution

$$\binom{q-(b-1)(p-1)+1}{p}. \tag{8b}$$

With $p=k$ and $q=n-k$ (8b) yields

$$\binom{n-bk+b}{k}, \tag{2b}$$

the number of k-combinations of the first n natural numbers such that if i occurs in a given combination, none of $i+1, i+2, \cdots, i+b$ can [**4**, p. 222].

For the circular case we get

$$\frac{p+q}{p+q-bp}\binom{q-(b-1)p}{p}, \tag{9b}$$

and

$$\frac{n}{n-bk}\binom{n-bk}{k}, \tag{5b}$$

[**4**, p. 222].

The other analogs are

$$\binom{k-1}{r-1}\binom{n-k-(b-1)(r-1)+1}{r}, \tag{4b}$$

(5b)
$$\frac{n}{n-k-(b-1)r}\binom{k-1}{r-1}\binom{n-k-(b-1)r}{r},$$

(6b)
$$\frac{n-k+r}{n-k-(b-1)r}\binom{k-1}{r-1}\binom{n-k-(b-1)r}{r},$$

and

(7b)
$$\frac{k-r}{n-k-(b-1)r}\binom{k-1}{r-1}\binom{n-k-(b-1)r}{r}.$$

In [1] the substitution $r=k-s$ in (4) was used to get Riordan's result [5] for the number of k-combinations of the first n natural numbers containing exactly s successions. For example, in 124569 the block 12 is a succession, the block 456 has the two successions 45 and 56, and the block 9 has none. Each block of length m contributes $m-1$ of these successions. It is clear that the same substitution in each of the above formulae on r blocks leads to an analogous interpretation in terms of successions.

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INTEGERS, NO THREE IN ARITHMETIC PROGRESSION

POU-SHUN CHIANG AND A. J. MACINTYRE, Western Michigan University and University of Cincinnati

Contrary to the assertion of Erdös and Turan (J. London Math. Soc., 11 (1936) 261–264, see page 263) there exist sets of 9 integers from 1–20 with no three in arithmetic progression. This fact was reported by a program run on the IBM 1620 at Wright-Patterson AFB.

Two such sets are:

(1)
$$1, 2, 6, 7, 9, 14, 15, 18, 20$$
$$1, 3, 6, 7, 12, 14, 15, 19, 20.$$

These are the only two of this kind. To establish this assertion, let us enumerate all sets of five integers selected from $1, \cdots, 10$ which contain no three in arithmetic progression. The sets may contain four members from 1 to 5 or four members from 6 to 10. The only two possibilities of this kind are:

(2)
$$\mathbf{1, 2, 4, 5, 10; \quad and \quad 1, 6, 7, 9, 10.}$$

$$\text{(5b)} \qquad \frac{n}{n-k-(b-1)r}\binom{k-1}{r-1}\binom{n-k-(b-1)r}{r},$$

$$\text{(6b)} \qquad \frac{n-k+r}{n-k-(b-1)r}\binom{k-1}{r-1}\binom{n-k-(b-1)r}{r},$$

and

$$\text{(7b)} \qquad \frac{k-r}{n-k-(b-1)r}\binom{k-1}{r-1}\binom{n-k-(b-1)r}{r}.$$

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$$\text{(2)} \qquad \mathbf{1, 2, 4, 5, 10;} \quad \textbf{and} \quad \mathbf{1, 6, 7, 9, 10.}$$

The sets may contain three members from 1 to 5, arranged in one of the following ways:

$$\begin{aligned} &1, 2, 4;\quad 1, 2, 5;\quad 1, 3, 4;\quad 1, 4, 5; \\ &2, 3, 5;\quad 2, 4, 5. \end{aligned} \tag{3}$$

It is easy to verify that these may be completed to sets of 5 from 1 to 10 only in the following ways:

$$\begin{aligned} &1, 2, 4, 8, 9;\quad 1, 2, 4, 8, 10;\quad 1, 2, 4, 9, 10 \\ &1, 2, 5, 7, 10 \\ &1, 3, 4, 6, 10;\quad 1, 3, 4, 8, 9;\quad 1, 3, 4, 8, 10;\quad 1, 3, 4, 9, 10 \\ &1, 4, 5, 8, 10 \\ &2, 3, 5, 9, 10 \\ &2, 4, 5, 9, 10 \end{aligned} \tag{4}$$

These sets taken in reverse order, that is explicitly the sets

$$\begin{aligned} &2, 3, 7, 9, 10;\quad 1, 3, 7, 9, 10;\quad 1, 2, 7, 9, 10 \\ &1, 4, 6, 9, 10 \\ &1, 5, 7, 8, 10;\quad 2, 3, 7, 8, 10;\quad 1, 3, 7, 8, 10;\quad 1, 2, 7, 8, 10 \\ &1, 3, 6, 7, 10 \\ &1, 2, 6, 8, 9 \\ &1, 2, 6, 7, 9 \end{aligned} \tag{5}$$

provide the complete tabulation of all sets of 5 from $1, \cdots, 10$ containing no three in arithmetic progression with two chosen from $1, \cdots, 5$ and three from $6, \cdots, 10$.

There are twenty-four sets of five numbers chosen from $1, \cdots, 10$ enumerated under (2), (4), and (5). It is easy to verify that none of them, with one exception, can be extended to sets of nine integers chosen from $1, \cdots, 20$ free of progressions of three terms. The exception is 1, 2, 6, 7, 9 which provides the first example of (1). The second example of (1) is consequently the only such set with four members in $1, \cdots, 10$ and five in $11, \cdots, 20$. No further possibilities remain. There cannot be six or more numbers included in $1, \cdots, 10$ or $11, \cdots, 20$.

One further comment on the paper quoted: the assertion made there, that at most 16 integers from $1, \cdots, 41$ can be selected free of an arithmetic progression of three terms requires reconsideration. It is however, only necessary to verify that neither set enumerated in (1) can be extended by eight more members of $22, \cdots, 41$ to complete the proof.

An independent determination of these results has been made by Mr. D. G. Burnell, using the computer facilities of the University of California at Davis.

For a discussion of many problems related to this one with extensive references to the literature, reference may be made to the lecture by P. Erdös published in *Lectures on Modern Mathematics*, Editor T. A. Saaty, Volume 3, Wiley, 1965. The lecture is titled *Some recent advances and current problems in number theory*, and appears on pages 196–244, with Section 7 and especially p. 223 most directly relevant.

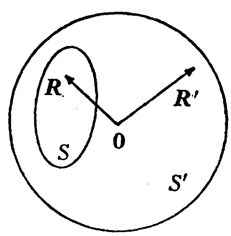
ON SOME PROBLEMS IN GRAVITATIONAL ATTRACTION

MURRAY S. KLAMKIN, Ford Motor Company, Dearborn Michigan

A rather interesting and at first sight surprising result is given by the following exercise (taken from A. S. Ramsey, An Introduction to the Theory of Newtonian Attraction, Cambridge University Press, Cambridge, 1961, p. 60, Ex. 7 and credited to a Cambridge college examination of 1905):

"Prove that the force required to separate the two parts of a sphere divided in any manner is $\gamma MM'\overline{GG'}/a^3$, where M, M' are the masses, and G, G' the centroids of the two parts and a is the radius of the sphere." (We are assuming the inverse square law of attraction between any two particles, i.e., $F=\gamma MM'/d^2$.)

We will first establish the preceding result together with an alternative equivalent expression. Using the equivalent expression, we will then show that the mutual force of attraction between the two subsets of the sphere is greatest when the two subsets are hemispheres. Additionally, we will consider some related problems of attraction for bodies other than spheres and for force laws other than inverse square.



Letting S denote one of the two subsets, then the other subset is the complement S'. The force of attraction between the two subsets is given by

$$F = \gamma \int_S \int_{S'} \frac{(R - R')}{|R - R'|^3} dVdV'$$

(For convenience, we have taken the density to be unity.) Since by Newton's third law (of action and reaction),

$$\int_S \int_S \frac{(R - R')}{|R - R'|^3} dVdV' = 0$$

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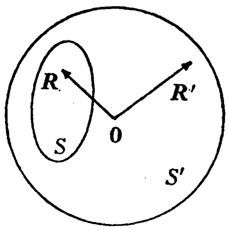

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(here R' is taken throughout S), F is also given by

$$F = \gamma \int_S \int_{\bar{S}} \frac{(R - R')}{|R - R'|^3} dV d\bar{V}$$

where $\bar{S}$ denotes the region of the entire sphere.

It is well known that the attraction between a sphere and a particle, when the latter is inside, varies directly as the radius vector from the particle to the center of the sphere. Thus,

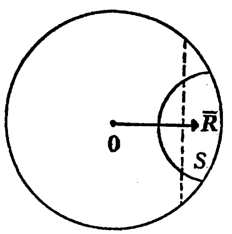
$$F = \frac{4\pi\gamma}{3} \int_S R dV$$

or $F = 4\pi\gamma M\bar{R}/3$ where M is the mass of S and $\bar{R}$ is the vector from the center of the sphere to the centroid of S. Since the centroid of $S+S'$ is at the center of the sphere, we also have the centrosymmetric result $F = -4\pi\gamma M\bar{R}''/3$. Then

$$F = \frac{4\pi\gamma}{3} \frac{(MM'R + MMR)}{M + M'} = \frac{\gamma MM'(R - R')}{a^3}.$$

The problem of maximizing $|F|$ is now seen to be equivalent to determining a subset S such that its moment about the center (in absolute value) is the greatest.

The more general problem of determining two subsets S and S' of a sphere with given masses such that the force of attraction between them is a maximum can be determined just as easily.

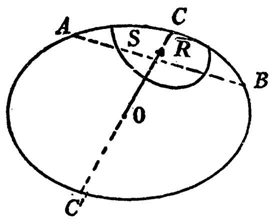


Let S denote the subset whose mass is $\leqq$ half the mass of the sphere. Now consider a plane perpendicular to the centroidal vector $\bar{R}$ of S which cuts off a segment of equal mass. Without loss of generality, we can assume that $\bar{R}$ terminates to the right of this plane (otherwise the moment of segment would be obviously greater). If any element dM of S lies to the left of this plane, its component in the direction of $\bar{R}$ will be less than a corresponding component of an element dM to the right of this plane and outside S. Consequently, $|F|$ is a maximum when S is a segment having the required mass. Also, by now varying the mass of S, it follows that the attraction between S and S' is greatest when S is a hemisphere.

Analogous results also hold when the sphere is replaced by an ellipsoid, i.e.,

$$\max_S \int_S R dV$$

occurs when S is a segment formed by a plane perpendicular to the major axis which cuts off the given mass. This will follow from the corresponding result for the sphere by using orthogonal projection. Under this transformation, the ratios of areas or volumes are preserved and centroids project into centroids. (For further properties and applications, see M. S. Klamkin and D. J. Newman, *The Philosophy and Applications of Transform Theory*, SIAM Review, Jan. 1961, pp. 12–14.) In order to simplify the discussion and the diagrams, however, we will establish the corresponding result for an ellipse.



Here S is an arbitrary set of given mass M and centroidal vector $\overline{\boldsymbol{R}}$. AB is a line parallel to the diameter conjugate to CC' such that the segment formed has the same mass M. It follows by orthogonal projection that the centroid of the segment also lies on the line OC. Using the same type of argument as before, we can show that the segment has a greater moment about O than S. Again by orthogonal projection, it follows that of all segments having a given area, the one formed by a line perpendicular to the major axis has its centroid at the greatest distance from O and consequently, has the greatest moment.

What we haven't proved and which apparently doesn't follow simply from the sphere result is the following conjecture:

If an ellipsoid is divided into two subsets such that the gravitational attraction between them is a maximum, then the two sets are congruent hemiellipsoids formed by a plane containing two of the axes.

Another series of related problems would be to determine a subset S, of given mass, of a given body such that

$$\boldsymbol{F} = \int_S \boldsymbol{G}(\boldsymbol{R})dV$$

is a maximum (in absolute value). Here $\boldsymbol{G}(\boldsymbol{R})$ is a specified vector function of the position vector $\boldsymbol{R}$ which originates from a fixed point of the body. Physically, this can be interpreted as determining a subset, of given mass, of a given body such that its attraction, due to a force law $\boldsymbol{G}(\boldsymbol{R})$, for a unit particle at a given position is a maximum.

Previously, we treated the case $\boldsymbol{G}(\boldsymbol{R}) = \boldsymbol{R}$. Another simple case occurs when $\boldsymbol{G}(\boldsymbol{R}) = \boldsymbol{R}/r$. If the body is a sphere and $\boldsymbol{R}$ originates from the center, then S will be a spherical sector having the given mass. Other cases will be treated in a subsequent paper.

Applying l'Hospital's rule

$$\lim_{u\to 0^+}\left(\frac{\sqrt{1+u+u^2}-1}{u}\right)=\lim_{u\to 0^+}\left(\frac{1+2u}{2\sqrt{1+u+u^2}}\right)=1/2.$$

(Quickies on page 166)

ON CURVES WITH CORNERS

JOHN H. STAIB, Drexel Institute of Technology

In an introduction to the subject of line integrals it is usually thought appropriate to simplify matters by restricting attention to a class of curves that is "sufficiently general for most applications." Most authors require that the parametric representation of the curve, say $P(t)$, $t\in[a, b]$, should be "piecewise smooth." An appropriate analytic definition for this term is given and it is indicated that the curve traversed by $P(t)$ will be "visually smooth" except, perhaps, for a finite number of "corners." It seems not to be noted, however, that such curves can be described by traversals that are continuously differentiable throughout their domain. This is suggested by the following example.

Example 1. Let

$$x(t)=\begin{cases}-t^2, & t<0\\ 0, & t\geqq 0\end{cases}\quad\text{and}\quad y(t)=\begin{cases}0, & t<0\\ t^2, & t\geqq 0.\end{cases}$$

Then the traversal $P(t)=(x(t), y(t))$, $t\in(-\infty, \infty)$, describes a curve C that has a 90° turn. (See Figure 1.) And yet $P(t)$ is continuously differentiable.

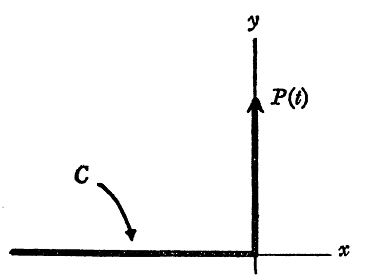


FIG. 1.

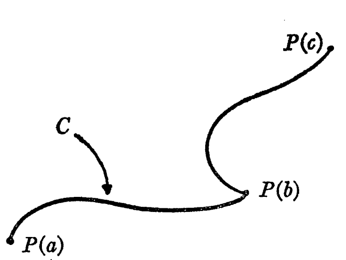


FIG. 2.

There is a "trick" to getting $P(t)$ around a corner without disturbing the continuity of its derivative: we must permit $P(t)$ to slow down to a halt as it rounds the corner. But this trick is always available. For suppose that a given $P(t)$ traverses the curve C shown in Figure 2 and that $P(t)$ is continuously differentiable except at $t=b$. Then we have only to replace $P(t)$ by a second traversal of C, say $Q(u)$, that has the special property that $Q'(\beta)=0$, where $Q(\beta)=P(b)$.

The idea is this: if $h(u)$ is a continuously differentiable, increasing function, with domain $[\alpha, \gamma]$ and range $[a, c]$, then $Q(u)=P(h(u))$, $u\in[\alpha, \gamma]$, is also a traversal of C—and in the same direction. We suppose that $b=h(\beta)$, so that $Q(\beta)=P(b)$. Then Q, in general, is misbehaved at β because P is misbehaved at b. But suppose that h has been chosen so that $h'(\beta)=0$. Then Q' not only exists at β but is continuous there. We see this as follows:

1. $$\frac{Q(\beta+\Delta u)-Q(\beta)}{\Delta u}=\frac{P(h(\beta+\Delta u))-P(h(\beta))}{\Delta u}$$
$$= P'(h(\beta)+\epsilon)\left\{\frac{h(\beta+\Delta u)-h(\beta)}{\Delta u}\right\}\to 0.$$

2. For $u\neq\beta$, we have $Q'(u)=P'(t)h'(u)$. But $\lim_{u\to\beta} h'(u)=0$. Therefore,
$$\lim_{u\to\beta} Q'(u)=0.$$

(In each of the foregoing arguments we have assumed that $P'(t)$ is bounded, but this is the case when we deal with "corners.") Thus, the construction of a continuously differentiable traversal for a curve with corners depends only on finding an appropriate function $h(u)$. But this is easy: we may take h as any function that rises from (α, a) to (γ, c) and has a zero slope at (β, b).

Example 2. Let

$$x(t)=\begin{cases} t,\ t\in[0,2]\\ -t+4,\ t\in(2,4]\end{cases}\quad\text{and}\quad y(t)=\begin{cases} t^2,\ t\in[0,1]\\ -t+2,\ t\in(1,2]\\ -t^2+5t-6,\ t\in(2,4].\end{cases}$$

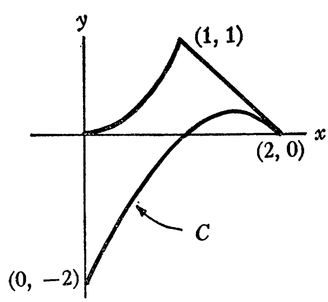


FIG. 3.

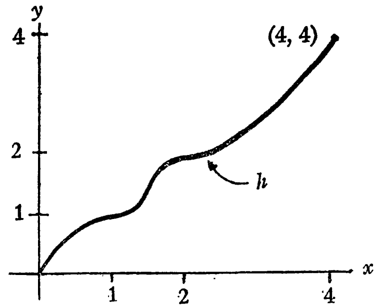


FIG. 4.

Then $P(t)=(x(t), y(t))$ describes the curve C shown in Figure 3. We require a function h that has the properties shown in Figure 4. The zero slope is important at $u=1$ and $u=2$, but we may have other points where the slope is zero. Thus in this case we do not even have to resort to a piecemeal description of h: we may use $h(u)=u-1/(2\pi)\sin 2\pi u$. Then $h'(u)=1-\cos 2\pi u$ and consequently, h is continuously differentiable in $[0, 4]$, is increasing with range $[0, 4]$, and has a zero slope at $u=1$ and $u=2$. Thus, we may take $Q(u)=P(h(u))$, $u\in[0, 4]$, as a continuously differentiable traversal of C.

DEFINITION 1. *Let C be the curve described by the traversal $P(t)$. Then a point $P(b)$ is said to be a corner-possible point of C revealed by $P(t)$ if either* (1) *$P'(t)$ is not continuous at b or* (2) $P'(b)=0$.

DEFINITION 2. *A curve C is said to be smooth if there exists a traversal of C that reveals no corner-possible points of C.*

DEFINITION 3. *A curve C is said to be piecewise smooth if there exists a traversal of C, say $P(t)$, such that* (1) *$P(t)$ reveals at most a finite number of corner-possible points and* (2) *$P'(t)$ is a bounded function.*

THEOREM. *If a curve C is piecewise smooth, then there is a traversal of C that is continuously differentiable throughout its domain.*

Proof. Let $P(t)$ be a traversal of C having the properties listed in Definition 3. Without loss of generality we may assume that the domain of $P(t)$ is $[0, 1]$. Also, because of proofs by mathematical induction, we may assume that $P(t)$ reveals exactly one corner-possible point at $t=b$. In this case we may take, where β is determined so that $b=h(\beta)$,

$$h(u) = \frac{1}{(1-\beta)^3+\beta^3}\left[(u-\beta)^3+\beta^3\right].$$

Then $h(0)=0$, $h(1)=1$, $h'(u)\geqq 0$, and $h'(\beta)=0$. It follows, by the reasoning used earlier, that $Q(u)=P(h(u))$ is the desired traversal.

ON A COMBINATORIAL PROOF

SADANAND VERMA, Western Michigan University

It is rather well known that the notion of the "Möbius function" and the Möbius inversion formula play an important role in number theory. A proof of the latter can be found in many number theory books. (See for example, [1] pp. 114–116). All of these proofs, besides converting a single sum into a double sum and then interchanging the order of the double sum depending on certain properties of the divisors, use several other properties of the Möbius function including the well known result $\sum_{d\mid n}\mu(d)=0$ or 1 according as $n>1$ or $n=1$.

In [2] it has been indicated that all this can be avoided for deducing

$$\phi(n) = n\prod_{p\mid n}\left(1-\frac{1}{p}\right) \quad \text{from} \quad \sum_{d\mid n}\phi(d) = n.$$

It should be observed that the strength of the method of proof used in [2] implies the Möbius inversion formula in the sense that it really includes a proof zero slope at $u=1$ and $u=2$. Thus, we may take $Q(u)=P(h(u))$, $u\in[0, 4]$, as a continuously differentiable traversal of C.

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It should be observed that the strength of the method of proof used in [2] implies the Möbius inversion formula in the sense that it really includes a proof of the same in its body without exhibiting it explicitly. As a result, we find the following combinatorial proof of the Möbius inversion formula.

THEOREM. *If* $F(n)=\sum_{d|n} f(d)$ *then* $f(n)=\sum_{d|n}\mu(d)F(n/d)$, *where* $f(n)$ *and* $F(n)$ *are numerical functions.*

Proof. Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ where the p_i are distinct primes and note that by definition $\mu(d)=0$ if d has a squared factor larger than one, that $\mu(1)=1$, and that $\mu(d)=(-1)^k$ if d is the product of k distinct primes. Let q_k run through all products of the distinct prime factors of n taken k at a time $(1\leqq k\leqq s)$. Thus we obtain

$$\sum_{d|n}\mu(d)F\left(\frac{n}{d}\right)=F(n)-\sum_{q_1}F\left(\frac{n}{q_1}\right)+\sum_{q_2}F\left(\frac{n}{q_2}\right)-\cdots$$
$$+(-1)^k\sum_{q_k}F\left(\frac{n}{q_k}\right)\cdots+(-1)^s\sum_{q_s}F\left(\frac{n}{q_s}\right).$$

Since $F(n)=\sum_{d|n}f(d)$, we get

$$\sum_{d|n}\mu(d)F\left(\frac{n}{d}\right)=\sum_{d|n}f(d)-\sum_{q_1}\sum_{d|n/q_1}f(d)+\sum_{q_2}\sum_{d|n/q_2}f(d)-\cdots \tag{1}$$
$$+(-1)^k\sum_{q_k}\sum_{d|n/q_k}f(d)\cdots+(-1)^s\sum_{d|n/q_s}f(d).$$

Obviously any term that appears in the single sum or a double sum on the right side of (1) arises out of a divisor d of n or out of a divisor d_k of $n/q_k(1\leqq k\leqq s)$, all of which are divisors of n as well. Let d be a fixed divisor of n other than n; then d must divide at least one of the numbers n/p_i $(i=1, 2, \cdots, s)$. Suppose it divides exactly ν of these numbers and let P_j run through all products, taken j at a time $(1\leqq j\leqq\nu)$, of only those primes p_i that occur in the denominators of the above ν of the numbers n/p_i $(i=1, 2, \cdots, s)$ that are divisible by the chosen divisor d. Then the specific divisor under consideration divides, in fact, all the numbers of the form n/P_j $(j=1, \cdots, \nu)$. Therefore, the chosen divisor d precipitates the term $f(d)$

$$\text{once in } \sum_{d|n}f(d)$$

$$-\binom{\nu}{1}\text{ times in } -\sum_{q_1}\sum_{d|n/q_1}f(d)$$

$$+\binom{\nu}{2}\text{ times in } +\sum_{q_2}\sum_{d|n/q_2}f(d)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(-1)^\nu\binom{\nu}{\nu}\text{ times in } (-1)^\nu\sum_{q_\nu}\sum_{d|n/q_\nu}\cdot$$

Thus the collected coefficient of the resulting term $f(d)$ on the right side of (1) is

$$1-\binom{\nu}{1}+\binom{\nu}{2}-\cdots+(-1)^k\binom{\nu}{k}\cdots+(-1)^\nu\binom{\nu}{\nu}=(1-1)^\nu=0.$$

In addition to all such divisors of n there is one and only one left, the number n itself. This divisor n precipitates the term $f(n)$ nowhere except once in $\sum_{d|n} f(d)$. Hence the value of the right side of (1) is simply $f(n)$ and this proves the theorem.

References

1. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill, New York, 1939.
2. S. Verma, A note on Euler's ϕ-function, this MAGAZINE, 38 (1965) 208–211.

ON CRITICAL GRAPHS OF DIAMETER 2

U. S. R. MURTY, University of Alberta

We consider unoriented graphs without loops and multiple edges. In such a graph the length of the shortest chain between two vertices is called the *distance* between them. The maximum of these distances taken over all pairs of vertices is called the *diameter* of the graph.

Let $m(n, s)$ denote the least number of edges that a graph on n vertices should possess in order that it be of diameter 2 and remain of diameter 2 even after dropping any s of the edges. (This property is equivalent to asking that between any two vertices in the graph there should at least be $s+1$ edge disjoint chains of length $\leqq 2$.) A graph on n vertices possessing this property and having $m(n, s)$ edges shall be called a *critical graph*. The object of this note is to characterize these critical graphs.

Let $\Gamma_n(s)$ denote the graph on n vertices ($n \geqq 2s+2$) obtained in the following way: the set of vertices is divided into two subsets A and B with $|A| = s+1$ and $|B| = n-s-1$. Any two vertices which do not simultaneously belong to B are joined by an edge. No two vertices in B are joined. It is easy to observe that this graph has

$$f(n, s) = (s+1)(n-s-1) + \frac{s(s+1)}{2}$$

edges and has the property described in the previous paragraph. Hence we have

$$m(n, s) \leqq f(n, s). \tag{1}$$

It can be verified that $m(4, 1) = f(4, 1)$ and that $\Gamma_4(1)$ is the corresponding critical graph.

THEOREM. *If* $n \geqq ((3+\sqrt{5})(s+1)/2)$, *then* $m(n, s) = f(n, s)$ *and* $\Gamma_n(s)$ *is the corresponding critical graph.*

Thus the collected coefficient of the resulting term $f(d)$ on the right side of (1) is

$$1-\binom{\nu}{1}+\binom{\nu}{2}-\cdots+(-1)^k\binom{\nu}{k}\cdots+(-1)^\nu\binom{\nu}{\nu}=(1-1)^\nu=0.$$

In addition to all such divisors of n there is one and only one left, the number n itself. This divisor n precipitates the term $f(n)$ nowhere except once in $\sum_{d|n} f(d)$. Hence the value of the right side of (1) is simply $f(n)$ and this proves the theorem.

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U. S. R. MURTY, University of Alberta

We consider unoriented graphs without loops and multiple edges. In such a graph the length of the shortest chain between two vertices is called the *distance* between them. The maximum of these distances taken over all pairs of vertices is called the *diameter* of the graph.

Let $m(n, s)$ denote the least number of edges that a graph on n vertices should possess in order that it be of diameter 2 and remain of diameter 2 even after dropping any s of the edges. (This property is equivalent to asking that between any two vertices in the graph there should at least be $s+1$ edge disjoint chains of length $\leqq 2$.) A graph on n vertices possessing this property and having $m(n, s)$ edges shall be called a *critical graph.* The object of this note is to characterize these critical graphs.

Let $\Gamma_n(s)$ denote the graph on n vertices ($n\geqq 2s+2$) obtained in the following way: the set of vertices is divided into two subsets A and B with $|A|=s+1$ and $|B|=n-s-1$. Any two vertices which do not simultaneously belong to B are joined by an edge. No two vertices in B are joined. It is easy to observe that this graph has

$$f(n, s) = (s+1)(n-s-1)+\frac{s(s+1)}{2}$$

edges and has the property described in the previous paragraph. Hence we have

$$m(n, s) \leqq f(n, s). \tag{1}$$

It can be verified that $m(4, 1)=f(4, 1)$ and that $\Gamma_4(1)$ is the corresponding critical graph.

THEOREM. *If $n\geqq((3+\sqrt{5})(s+1)/2)$, then $m(n, s)=f(n, s)$ and $\Gamma_n(s)$ is the corresponding critical graph.*

Proof. Consider a critical graph. Every vertex of this graph has to be of degree $\geqq s+1$. We shall prove that this graph will have a vertex of degree equal to $s+1$. If there is no vertex of degree $s+1$ suppose that $d>s+1$ is the minimum degree. Let x be a vertex of degree d. Let $\Gamma x=\{x_1, x_2, \cdots, x_d\}$ be the set of vertices adjacent to x and let Y denote the set of vertices different from $x, x_1, x_2, \cdots, x_d$. The number of edges between x and Γx is d. Each vertex in Γx will be joined to at least s other vertices in Γx and this would account for at least $(ds)/2$ edges. Each of the vertices in Y will be joined to at least $s+1$ vertices in Γx accounting for another $(n-d-1)(s+1)$ edges. The degree of each vertex in Y is at least d and this will account for at least another $(n-d-1)(d-s-1)/2$ edges. Hence the number of edges in the critical graph will be at least

$$m+\frac{ds}{2}+(n-d-1)(s+1)+\frac{(n-d-1)(d-s-1)}{2}=a(d, n, s) \text{ (say).}$$

Also we have

$$a(n, d, s) \leqq m(n, s) \leqq f(n, s). \tag{2}$$

Simplifying (2) we have

$$d \geqq n-s-1. \tag{3}$$

Again, as each of the vertices in the critical graph has degree $\geqq d$, it must have at least $(nd)/2$ edges and therefore

$$\frac{nd}{2} \leqq m(n, s) \leqq f(n, s). \tag{4}$$

From (4) we have

$$d \leqq \frac{(s+1)(2n-s-2)}{2}. \tag{5}$$

Combining (3) and (5) we have

$$n-s-1 \leqq d \leqq \frac{(s+1)(2n-s-2)}{n}$$

or

$$n^2-n(3s+3)+s^2+3s+2 \leqq 0. \tag{6}$$

From (6) it follows that

$$n<\frac{(3+\sqrt{5})(s+1)}{2}. \tag{7}$$

But (7) contradicts the hypothesis of the theorem and therefore, under the hypothesis of the theorem, every critical graph has a vertex of degree $s+1$.

Now if x is a vertex of degree $s+1$ in a critical graph, the vertices in Γx will have to be joined to each other, and each of the vertices different from the vertices in Γx will have to be joined to all the vertices in Γx. The graph does not need any further edges to possess the desired property and none of these edges can be deleted. This means that $\Gamma_n(s)$ is essentially the only critical graph. This completes the proof of the theorem.

Remark. For the case $s=1$, a slightly better use of (6) than in (7) will show that the $\Gamma_n(1)$ are the critical graphs for $n \geqq 5$. We have already observed that this statement is true for $n=4$, so $\Gamma_n(1)$ are the critical graphs for all $n \geqq 4$. In general it appears true that $\Gamma_n(s)$ will be the critical graphs for all $n \geqq 2s+2$, but it has not been possible to prove this by using the present method.

The author wishes to thank the referee whose comments led to an improvement in the presentation of the paper.

A NOTE ON COMPLEX POLYNOMIALS HAVING ROLLE'S PROPERTY AND THE MEAN VALUE PROPERTY FOR DERIVATIVES

W. G. DOTSON, JR., North Carolina State University

It is well known that Rolle's Theorem and the mean value theorem for derivatives do not hold for complex-valued functions on the real line, e.g., $f(x)=x(x-1)e^{ix}$ vanishes at $x=0$ and $x=1$, but $f'(x)=e^{ix}[(2x-1)+i(x^2-x)]$ does not vanish for any x in $(0, 1)$. This is a special case of the (also well known) fact that precise analogues of Rolle's Theorem and the mean value theorem do not hold for analytic functions of a complex variable, e.g., the polynomial $p(z)=z^3-1$ vanishes at 1 and $\omega=(-1/2)+i(\sqrt{3}/2)$, but $p'(z)=3z^2$ does not vanish at any point on the linear segment from 1 to ω. Of course, if an analytic function $F(z)$ maps the real line into itself and has only real zeros, then Rolle's Theorem will hold for $F(z)$—in the sense that the linear segment between any two zeros of $F(z)$ will contain a zero of $F'(z)$. Notice that $p(z)=z^3-1$ maps the real line into itself but does not have only real zeros, whereas $f(z)=z(z-1)e^{iz}$ has only real zeros but does not map the real line into itself. Notice, also, that Rolle's Theorem holding for $F(z)$ does not imply that the mean value theorem holds for $F(z)$, e.g., Rolle's Theorem holds for $F(z)=\sin z$, but $[\sin(i+2\pi)-\sin(i)]/2\pi=0$ and $F'(z)=\cos z$ does not vanish at any point of the linear segment from i to $i+2\pi$.

There exists an extensive literature (see [**1**] and the bibliography of [**1**]) devoted to establishing analogues or generalizations of Rolle's Theorem which do hold for *polynomials* in a complex variable ([**1**], p. 21). The simplest of these is the famous Gauss-Lucas Theorem which says that if $P(z)$ is a complex polynomial then the zeros of $P'(z)$ all lie in the convex hull of the set of zeros of $P(z)$. For an elementary proof of this, the reader is referred to [2, p. 84]. There are many generalizations of the Gauss-Lucas Theorem (see [**3**], pp. 202, 210 and [**1**]), and there are many other theorems which describe the location of the zeros of $P'(z)$ relative to the zeros of the polynomial $P(z)$. As interesting examples, tices in Γx will have to be joined to all the vertices in Γx. The graph does not need any further edges to possess the desired property and none of these edges can be deleted. This means that $\Gamma_n(s)$ is essentially the only critical graph. This completes the proof of the theorem.

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If $P(z)$ is an nth degree polynomial ($n>1$) and if $P(a)=P(b)=0$ (where $a\neq b$), then $P'(z)$ has at least one zero in the region comprised of all points from which the line segment ab subtends an angle of at least $\pi/(n-1)$.

We note that analogues of the mean value theorem for derivatives can be obtained from Fekete's Theorem and from the Grace-Heawood Theorem by the simple device used in real analysis to prove the mean value theorem from Rolle's Theorem. For suppose $Q(z)$ is a polynomial of degree $n>1$ and $a\neq b$. Then $P(z)=Q(z)-Q(a)-[(Q(b)-Q(a))/(b-a)](z-a)$ is a polynomial of degree $n>1$ and $P(a)=P(b)=0$. Hence

$$P'(z) = Q'(z) - [(Q(b) - Q(a))/(b - a)]$$

vanishes at least once in the region prescribed by Fekete's Theorem (and it vanishes at least once in the circular region prescribed by the Grace-Heawood Theorem). Marden [**1**, p. 110], has published more general analogues (for complex polynomials) of the mean value theorem. Results have also been obtained concerning the location of the zeros of $f'(z)$ relative to the zeros of $f(z)$ for certain classes of transcendental entire functions $f(z)$ having real zeros (see [**3**, p. 201]).

We shall ask some slightly different questions which do not appear to have been answered explicitly in the literature. The answers are established with elementary proofs, which, along with the above referenced elementary proof of the Gauss-Lucas Theorem, could be presented to beginning classes in complex analysis. The following definitions are used:

DEFINITION 1. *An entire function $f(z)$ is said to have Rolle's property provided that if z_1 and z_2 are any two zeros of $f(z)$, then there exists a point z between z_1 and z_2 (i.e., z lies on the linear segment from z_1 to z_2, and $(z-z_1)(z-z_2)\neq 0$) such that $f'(z)=0$.*

DEFINITION 2. *An entire function $f(z)$ is said to have the mean value property for derivatives provided that if z_1 and z_2 are any two points in the complex plane, then there exists a point z between z_1 and z_2 such that $f(z_2)-f(z_1)=(z_2-z_1)f'(z)$.*

THEOREM 1. *A complex polynomial $P(z)$ has Rolle's property if and only if its zeros are collinear.*

Proof. First, suppose the zeros of $P(z)$ lie on some line L. The fact that $P(z)$ must then have Rolle's property is, essentially, a special case of an exercise proposed by Marden [**1**, p. 45, Exercise 1]. By means of a translation and rotation, $w=az+b$, $|a|=1$, one maps L onto the real line, R. The polynomial $Q(w)=P([w-b]/a)$ then has distinct real zeros $w_j=az_j+b$, $j=1,\cdots,k$, where z_j, $j=1,\cdots,k$, are the distinct zeros of $P(z)$ (all on L). Hence

$$Q(w) = c_0 \prod_{j=1}^{k} (w - w_j)^{n_j},$$

where $c_0 \neq 0$ and n_j = the order of w_j; and so $\bar{c}_0 Q(w)$ maps R into R and has zeros only on R. Thus $\bar{c}_0 Q(w)$ has Rolle's property. But $\{\bar{c}_0 Q(w)\}' = (\bar{c}_0/a) \cdot P'([w-b]/a) = (\bar{c}_0/a)P'(z)$; and if $w_i < w < w_j$ then, under rotation and translation, $z = [w-b]/a$ lies between $z_i = [w_i - b]/a$ and $z_j = [w_j - b]/a$ on L. Hence $P(z)$ has Rolle's property.

Conversely, suppose $P(z)$ is a polynomial of degree N which has Rolle's property. If the zeros of $P(z)$ are not collinear, then some three must form a triangle. Since the set of all triangles formed by the zeros is finite, there must be at least one such triangle $\triangle$ which has area at least as small as the area of any other such triangle. Denote the vertices of $\triangle$ by z_1, z_2, z_3, and the remaining distinct zeros (if any) of $P(z)$ by $z_4, \cdots, z_k$. It is clear that for $j \geqq 4$, z_j cannot lie either inside or on the boundary of $\triangle$. Furthermore, each z_j ($j \geqq 4$) can be joined with a straight line segment to some vertex z_1, z_2, z_3 of $\triangle$ in such a way that (1) no two segments emanating from different vertices of $\triangle$ intersect, and (2) each segment intersects the closure of $\triangle$ only at one vertex of $\triangle$. (See Figure 1.)

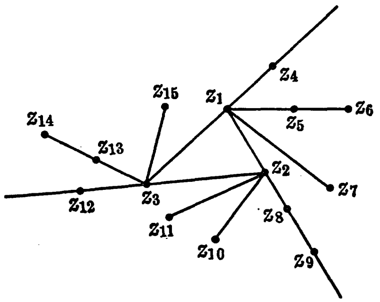

FIG. 1

Since the segment between any two zeros of $P(z)$ must contain a zero of $P'(z)$, one finds that there must be at least $k-3$ zeros of $P'(z)$ all of which are exterior to the closure of $\triangle$ and none of which coincides with any zero of $P(z)$. (It does not matter if several zeros of $P(z)$ are collinear with one vertex of $\triangle$.) The three segments between the vertices of $\triangle$ yield three more zeros of $P'(z)$ distinct from the zeros of $P(z)$. Let n_j = the order of the zero z_j of $P(z)$, $j = 1, 2, \cdots, k$; so that

$$\sum_{j=1}^{k} n_j = N.$$

Then $P'(z)$ has a zero of order $n_j - 1$ at z_j (if $n_j > 1$), $j = 1, 2, \cdots, k$. Finally, then, the degree of $P'(z)$, viz., $N-1$, cannot be less than

$$\sum_{j=1}^{k} (n_j - 1) + (k - 3) + (3) = (N - k) + (k - 3) + (3) = N.$$

This contradiction completes the proof that the zeros of $P(z)$ must be collinear.

We note that this proof could be shortened by reference to a recent result of Gemignani, [4], that any finite set of noncollinear points in the plane forms the set of vertices of at least one simple closed polygonal path. However, the proof of this result is rather long and intricate, and so we prefer our above argument in the interest of basic simplicity.

THEOREM 2. *A complex polynomial $P(z)$ has the mean value property for derivatives if and only if the degree of $P(z)$ does not exceed two.*

Proof. If $P(z)=az^2+bz+c$ then

$$P(z_2) - P(z_1) = (z_2 - z_1)P'\left(\frac{z_1 + z_2}{2}\right)$$

is easily seen to be an identity in z_1, z_2. Conversely, suppose $P(z)$ has the mean value property for derivatives, and suppose the degree of $P(z)$ is $N>2$. Then $P(z)$ has Rolle's property, and so, by Theorem 1, the distinct zeros $z_1, \cdots, z_k$ of $P(z)$ lie on a line L. The $k-1$ segments between these z_j must contain $k-1$ zeros of $P'(z)$ distinct from each other and from the z_j; and, as seen above, a total of $N-k$ zeros of $P'(z)$ (each counted with its multiplicity) coincide with the z_j's. Hence all $N-1=(k-1)+(N-k)$ zeros of $P'(z)$ must lie on L (a result also obtainable from the Gauss-Lucas Theorem). Now suppose c is any complex number. Let $w_1, \cdots, w_r$ be the distinct zeros of $P(z)-c$, with orders $m_1, \cdots, m_r$ respectively, where $\sum_{i=1}^{r} m_i=N>2$. Then $P(w_i)=c$ for each $i=1, \cdots, r$, and so we have $P(w_i)-P(w_j)=c-c=0$ for all $i, j=1, \cdots, r$; hence, for $i\neq j$, the segment from w_i to w_j must contain a zero of $P'(z)$. Suppose now that, for some i, w_i lies off the line L. Then no w_j can lie on L, for there would be no zero of $P'(z)$ between w_i and a w_j on L. Hence, all w_j's lie off L. But there can be at most one of the w_j's on each side of the line L (since there would be no zero of $P'(z)$ between two w_j's on the same side of L). There are, then, two cases: (1) w_i, lying on one side of L, is the only distinct zero of $P(z)-c$, and (2) there are only two distinct zeros w_i, w_j of $P(z)-c$, and these lie on opposite sides of L. In case (1), $m_i=N>2$ and so $P'(z)=[P(z)-c]'$ has a zero of order $m_i-1>1$ at w_i. This is a contradiction, since all zeros of $P'(z)$ lie on L. In case (2), we have $m_i+m_j=N>2$, and so either $m_i\geqq 2$ or $m_j\geqq 2$. Hence $P'(z)$ must have a zero either at w_i or w_j, and this again is a contradiction. It follows that $w_1, \cdots, w_r$ must all lie on the line L. But $w_1, \cdots, w_r$ are all of the distinct pre-images of c under the mapping $P(z)$, and c was an arbitrary complex number. Thus the domain of $P(z)$ is contained in the line L, which is our final contradiction and completes the proof of the theorem.

There remains the interesting question as to whether or not there exists a transcendental entire function which has the mean value property for derivatives. There are, of course, transcendental entire functions having Rolle's property, e.g., $F(z)=\sin z$, but we conjecture that there are no such functions having the mean value property for derivatives.

We conclude with a simple analogue of the mean value theorem for derivatives which is valid for any analytic function of a complex variable. This result seems to have escaped attention in the literature, perhaps because of its simplicity.

THEOREM 3. *If $f(z)$ is analytic in a region R containing points z_1, z_2 together with the line segment between them, then there exist points z', z'' on this line segment such that*

$$f(z_2) - f(z_1) = (z_2 - z_1)\{Ref'(z') + iImf'(z'')\}.$$

Proof. With $z(t) = z_1 + t(z_2 - z_1)$, $0 \leqq t \leqq 1$, we have

$$\int_0^1 f'(z(t))(z_2 - z_1)dt = f(z_2) - f(z_1).$$

Dividing through by $z_2 - z_1$ and separating the resulting equation into real and imaginary parts, we find that the theorem follows by application of the mean value theorem for integrals of continuous real functions to the integrals

$$\int_0^1 Ref'(z(t))dt \quad and \quad \int_0^1 Imf'(z(t))dt.$$

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A NOTE ON A FORM OF RATIO TEST

S. H. L. KUNG, Jacksonville University

Knopp [1] has given the following criterion for testing convergence of a positive infinite series:

Let $f(x)$ be a positive, continuous, and monotone-decreasing function and $g(x)$ be a differentiable, monotone-increasing function such that $g(x) > x$, and suppose that

$$\lim_{x \to \infty} \frac{g'(x)f(g(x))}{f(x)} = r. \tag{1}$$

Then the infinite series $\sum_1^\infty a_n = \sum_1^\infty f(n)$ converges if $r < 1$ and diverges if $r > 1$.

In this note we give a more general proof for the above criterion and interpret its results.

THEOREM. *Let $f(x)$ be a positive, continuous, and monotone-decreasing function, and $g(x)$ and $h(x)$ be differentiable, monotone-increasing functions such that*

$$\lim_{x \to \infty} g(x) = \infty, \tag{2a}$$

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$$\lim_{x\to\infty} g(x) = \infty, \tag{2a}$$

(2b) $$\lim_{x\to\infty} h(x) = \infty,$$

(2c) $$g(x) > h(x),\ h'(x) \neq 0, \quad \text{for } x > 0;$$

and suppose that

(3) $$\lim_{x\to\infty} \frac{g'(x)f(g(x))}{h'(x)f(h(x))} = r.$$

Then the infinite series $\sum_1^\infty a_n = \sum_1^\infty f(n)$ *converges if* $r<1$ *and diverges if* $r>1$.

Proof. (i) For $r<1$, there exists $\epsilon>0$ such that for sufficiently large c and $x>c$

(4) $$\frac{g'(x)f(g(x))}{h'(x)f(h(x))} = r + \epsilon < 1.$$

So

$$\int_c^I g'(x)f(g(x))dx < (r+\epsilon)\int_c^I h'(x)f(h(x))dx, \qquad I > c,$$

or

(5) $$\int_t^v f(x)dx < (r+\epsilon)\int_s^u f(x)dx$$

where $t=g(c)$, $v=g(I)$, $s=h(c)$, and $u=h(I)$.

Using (5) and noting from (2c) that $v>u$, $t>s$, we have

(6) $$\begin{aligned}&\int_s^t f(x)dx - \int_u^v f(x)dx \\ &\quad = \int_s^u f(x)dx - \int_t^v f(x)dx > (1-r-\epsilon)\int_s^u f(x)dx > 0.\end{aligned}$$

Therefore, $\int_u^v f(x)dx$ or $\int_{h(I)}^{g(I)} f(x)dx$ is bounded. Although the boundedness of $\int_{h(I)}^{g(I)} f(x)dx$ is not sufficient in general for the boundedness of $\int_1^\infty f(x)dx$, in this case if $\int_{h(I)}^{g(I)} f(x)dx$ is bounded by (6), then $\int_{h(c)}^{h(I)} f(x)dx$ is bounded and this implies that $\int_1^\infty f(x)dx$ is bounded. So $\int_1^\infty f(x)dx$ converges and, by Cauchy's integral test, $\sum_1^\infty a_n = \sum_1^\infty f(n)$ converges.

(ii) For $r>1$, there exists $\epsilon>0$ such that when $x>c$

(7) $$\frac{g'(x)f(g(x))}{h'(x)f(h(x))} = r - \epsilon > 1.$$

Integrating (7), we have

(8) $$\int_t^v f(x)dx > (r-\epsilon)\int_s^u f(x)dx.$$

It follows, then, that

$$\begin{aligned}(9)\quad &\int_u^v f(x)dx - \int_s^t f(x)dx \\ &= \int_t^v f(x)dx - \int_s^u f(x)dx > (r - \epsilon - 1)\int_s^u f(x)dx > 0.\end{aligned}$$

Therefore, $\int_u^v f(x)dx$ or $\int_{h(l)}^{g(l)} f(x)dx$ is unbounded and hence $\int_1^\infty f(x)dx$ and $\sum_1^\infty a_n = \sum_1^\infty f(n)$ diverge.

We obtain Knopp's criterion by letting $h(x)=x$ in (3). If $g(x)=e^x$, $h(x)=x$, (3) becomes the form of Ermakov's ratio test [1, 2]. If $g(x)=x+1$, $h(x)=x$, then $r=\text{Lim}_{x\to\infty} f(x+1)/f(x)$ which bears some resemblance to the usual form of d'Alembert's ratio test.

Example 1. Prove that the series $\sum_1^\infty 1/n$ diverges.

Choose $f(x)=1/x$, $g(x)=e^x$, $h(x)=x$, $x>0$. From (3) $r=\text{Lim}_{x\to\infty} x>1$. So, the series diverges.

Example 2. Establish convergence or divergence of the series $\sum_2^\infty \log n/n^p$.

Choose $f(x)=\log x/x^p$, $g(x)=x^2$, $h(x)=x$, $x>1$. From (3) we have

$$\operatorname*{Lim}_{x\to\infty}\left[\frac{2\log x^2}{x^{2p}}\cdot\frac{x^{p+1}}{\log x}\right] = 4 \operatorname*{Lim}_{x\to\infty} x^{1-p} = r.$$

If $p>1$, $r=0$; if $0<p\leqq 1$, $r>1$. Hence, $\sum_2^\infty \log n/n^p$ converges if $p>1$ and diverges if $0<p\leqq 1$.

Example 3. Test the convergence of the series $\sum_3^\infty 1/n \log n(\log\log n)^p$.

Choose $f(x)=1/x \log x(\log\log x)^p$, $g(x)=2^{2^{x^2}}$, $h(x)=2^{2^x}$, $x\geqq 3$. From (3) we have

$$\operatorname*{Lim}_{x\to\infty}\left[x2^{(2^{x^2}+x^2-x-2^x+1)}\frac{2^{2^x}\log 2^{2^x}(\log\log 2^{2^x})^p}{2^{2^{x^2}}\log 2^{2^{x^2}}(\log\log 2^{2^{x^2}})^p}\right] = 2\operatorname*{Lim}_{x\to\infty} x^{1-p} = r.$$

If $p>1$, $r=0$; hence the series $\sum_3^\infty 1/n \log n(\log\log n)^p$ converges if $p>1$. More generally, it may be shown that $\sum_K^\infty 1/n(\log n)(\log\log n)\cdots(\log\log\cdots\log n)^p$ (where K is sufficiently large for $(\log\log\cdots\log n)$ to be defined) converges if $p>1$.

Many other infinite series whose convergence or divergence is usually established by either the integral test or d'Alembert's test may be investigated by using (3).

The author would like to thank the reviewer for his valuable suggestions.

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A NOTE ON ONE-SIDED DIRECTIONAL DERIVATIVES

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The notions of *one-sided* derivatives and directional derivatives are usually passed over rather quickly in calculus. The student hardly gets a chance to see any of the uses or applications of these notions. Also when the subject of extrema of functions comes up, the student is often told that if the function is not differentiable, the problem is not open to attack by the tools of calculus. As a result, he feels completely unarmed when faced with extremal problems involving functions which are not differentiable. Fortunately, however, many extremal problems in analysis and its applications involve functions which have one-sided directional derivatives, and may even be convex. In such problems, one-sided directional derivatives play an important role in characterizing and seeking extrema. The purpose of the present note is to examine this role.

Throughout our discussion, R will always denote the set of all real numbers, E a linear space, and S a subset of E.

A function $f\colon S \to R$ is said to have a *one-sided* directional derivative at a point $x_0 \in S$ if for each fixed $h \in E$ for which $x_0 + th \in S$ for sufficiently small positive t,

$$D^+f(x_0;h) = \lim_{t\to 0^+} \frac{f(x_0+th)-f(x_0)}{t} \tag{1}$$

exists in the extended real numbers. $D^+f(x_0;\, h)$ is also called the one-sided Gateaux variation. It is easy to show that $D^+f(x_0;\, h)$ is positively homogeneous of degree one in h, i.e.,

$$D^+f(x_0;\,\tau h) = \tau D^+f(x_0;\, h) \quad \text{for } \tau > 0. \tag{2}$$

The following proposition is an immediate consequence of (1) and (2).

PROPOSITION 1. *A necessary and sufficient condition for f to have a one-sided directional derivative at $x_0 \in S$ is that the following representation holds for each h for which $x_0+th \in S$ for $0 \leqq t \leqq 1$:*

$$f(x_0+h) - f(x_0) = \alpha(x_0;\, h) + r(x_0;\, h), \tag{3}$$

where

(4) $\alpha(x_0;\, h)$ *is positively homogeneous of degree one in h,*

and

$$\lim_{t\to 0^+} \frac{r(x_0;\, th)}{t} = 0. \tag{5}$$

We note that if a representation such as (3) exists, with properties (4) and (5), then the representation is unique and $\alpha(x_0;\, h) = D^+f(x_0;\, h)$.

Remark 1. In the case of a function of a real variable $D^+f(x_0;\, h)$ should be clearly distinguished from the one-sided derivatives

$$f'_+(x_0) = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

and

$$f'_-(x_0) = \lim_{\Delta x \to 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

It is easy to show in this case that the one-sided directional derivative at an interior point x_0 exists if and only if both $f'_+(x_0)$ and $f'_-(x_0)$ exist; moreover,

$$D^+f(x_0; h) = \begin{cases} f'_+(x_0)h, & h > 0, \\ f'_-(x_0)h, & h < 0. \end{cases}$$

For example, if $f(x) = |x|$, then

$$D^+f(a; h) = \begin{cases} \dfrac{ah}{|a|}, & a \neq 0 \\ |h|, & a = 0. \end{cases}$$

The one-sided derivatives should also be carefully distinguished from the right-hand limit of the derivative. (See [4].)

PROPOSITION 2. *Let f: $S \to R$ have a one-sided directional derivative at a point $x_0 \in S$. A necessary condition for f to have a relative minimum at x_0 is that*

$$D^+f(x_0; x - x_0) \geqq 0$$

for all $x \in S$ for which $D^+f(x_0; x - x_0)$ exists.

Proof. We prove the contrapositive of the theorem. Suppose $D^+f(x_0; u - x_0) < 0$ at some $u \in S$. Then by Proposition 1, letting $h = u - x_0$, we have

$$f(x_0 + th) - f(x_0) = tD^+f(x_0; h) + r(x_0; th) \tag{6}$$

for sufficiently small positive t, where

$$\lim_{t \to 0^+} \frac{r(x_0; th)}{t} = 0.$$

Thus since $D^+f(x_0; h) < 0$, the right-hand side of (6) can be made negative for all sufficiently small positive t. Hence f does not have a relative minimum at x_0.

For convex functions, one-sided directional derivatives can be used to *characterize* an absolute minimum. Recall that a nonempty set K in a linear space E is said to be convex if $tx + (1-t)y \in K$ for each $x, y \in K$ and $0 \leqq t \leqq 1$. A function ϕ: $K \to R$ is called *convex* if

$$\phi(tx + (1 - t)y) \leqq t\phi(x) + (1 - t)\phi(y)$$

for all $x, y \in K$ and $0 \leqq t \leqq 1$.

It is not hard to show that if ϕ is a convex function, then for each fixed x_0, $x \in K$, $t^{-1}[\phi(x_0 + t(x - x_0)) - \phi(x_0)]$ is an increasing function of t for $t > 0$. (See for example [**1–3**] 5.) Thus

$$D^+\phi(x_0; x - x_0) = \lim_{t\to 0^+} \frac{\phi(x_0 + t(x - x_0)) - \phi(x_0)}{t}$$

exists in the extended real numbers and is equal to

$$\inf\{t^{-1}[\phi(x_0 + t(x - x_0)) - \phi(x_0)]: t > 0\}.$$

PROPOSITION 3. *A necessary and sufficient condition for a convex function* ϕ *defined on a convex set* K *to have an absolute minimum at* $x_0 \in K$ *is that for all* $x \in K$,

$$D^+\phi(x_0; x - x_0) \geqq 0. \tag{7}$$

Proof. From the remark preceding the statement of this proposition, it follows that

$$t^{-1}[\phi(x_0 + t(x - x_0)) - \phi(x_0)] \geqq D^+\phi(x_0; x - x_0).$$

In particular, for $t = 1$,

$$\phi(x) - \phi(x_0) \geqq D^+\phi(x_0; x - x_0)$$

which establishes the sufficiency of condition (7). The necessity was established in Proposition 2.

As an illustration of Proposition 3, we consider the following application.

Let K be a convex set in a real or complex Euclidean space H with inner product (x, y) and norm $\|x\| = \sqrt{(x, x)}$. Let y be a fixed point in H, $y \notin K$. A point $x_0 \in K$ is said to be a closest point to y if

$$\|x_0 - y\| \leqq \|x - y\|$$

for all $x \in K$. We now use condition (7) to recover the known characterization of closest points in this case, namely, x_0 is a closest point to y if and only if

$$\mathrm{Re}(y - x_0, x_0 - x) \geqq 0 \tag{8}$$

for all $x \in K$, where Re z denotes the real part of z.

Define $\phi(x) = \|x - y\|$. Then ϕ is convex and hence, by Proposition 3, a necessary and sufficient condition for x_0 to minimize $\phi(x)$ is that $D^+\phi(x_0; x - x_0) \geqq 0$ for all $x \in K$. But

$$\begin{aligned} D^+\phi(x_0; x - x_0) &= \lim_{t\to 0^+} \frac{\|x_0 + t(x - x_0) - y\| - \|x_0 - y\|}{t} \\ &= \lim_{t\to 0^+} \frac{2t\,\mathrm{Re}(x_0 - y, x - x_0) + t^2\|x_0 - y\|^2}{t(\|x_0 + t(x - x_0) - y\| + \|x_0 - y\|)} \\ &= \frac{\mathrm{Re}(x_0 - y, x - x_0)}{\|x_0 - y\|} \end{aligned}$$

from which the characterization (8) follows.

Remark 2. The study of convex functions is gradually being introduced in calculus books. Elegant presentations of elementary properties of convex functions may be found in the books by Hille [3] and Fleming [2], in the introduction to Artin's little classic on the gamma function [1] and in an early paper by Taylor [5]. While differentiability properties of convex functions are discussed in these and other references, the presentations stop short of the characterization provided in Proposition 3.

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CANONICAL PLACEMENT OF SIMPLICES

JOHN W. KENELLY and ANDREW SOBCZYK, Clemson University

With respect to an orthonormal basis in Euclidean space E^{n+1}, the $n+1$ unit points are the vertices of an equilateral n-simplex. This suggests the question, "Is an arbitrary n-simplex congruent to a simplex with vertices of the form $(0, 0, \cdots, 0, a_i, 0, \cdots, 0)$, $i=0, \cdots, n$?" The answer in general is "no," since it is evident that a triangle containing an obtuse angle is not so placeable in E^3. The answer for $n \geqq 3$ is unexpected. An n-simplex is congruent to a simplex with vertices on the coordinate axes of E^{n+1} if and only if for each vertex, the pairs of edge vectors originating at the vertex have equal positive inner products. In case $n=2$, any acute angled triangle is placeable with vertices on the coordinate axes of E^3. The n vectors originating at a vertex, v_i, of an n-simplex, and terminating at the n adjacent vertices, will be referred to as "co-original edge vectors," e.g., v_{ij} is the vector extending from vertex v_i to vertex v_j.

Investigation of this question is motivated in part by the following consideration. The placeability of a simplex on the coordinate axes is equivalent to the existence of a point, from which vectors to the vertices of the simplex are mutually orthogonal.

LEMMA 1. *A triangle may be placed with its vertices on the positive coordinate axes of E^3 if and only if the interior angles of the triangle are acute.*

Proof. Let $\phi_i = (\delta_{i0}, \delta_{i1}, \delta_{i2})$, $i=0, 1, 2$, be the standard orthonormal vectors along the rectangular coordinate axes. A triangle with vertices $a_0\,\phi_0$, $a_1\,\phi_1$, $a_2\,\phi_2$, $a_j \neq 0$, $j=0, 1, 2$, then has its lengths of sides a, b, c, given by $a^2=a_1^2+a_2^2$, $b^2=a_0^2+a_2^2$, $c^2=a_0^2+a_1^2$. The solutions for a_0^2, a_1^2, a_2^2 of the preceding equations are $a_0^2=(c^2+b^2-a^2)/2$, $a_1^2=(a^2+c^2-b^2)/2$, $a_2^2=(a^2+b^2-c^2)/2$. These are positive; thus the law of cosines gives the acute angle conclusion. Conversely, given any triangle in which the angles are acute, the preceding equations determine a_j, $j=0, 1, 2$, to place the triangle.

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LEMMA 2. *In the class of Euclidean n-simplices, the inner product values of pairs of co-original edge vectors at one vertex uniquely determine the inner product values at all of the other vertices. In particular, if they are constant at one vertex, they must be constant at each individual one of the other vertices, and at most one vertex can have an associated nonpositive value.*

Proof. Suppose that the $n+1$ vertices of the n-simplex have vector positions $v_j, 0 \leqq j \leqq n$ and that the inner product values of pairs of co-original edge vectors at v_0 are known. Without loss of generality, we assume that the vertex v_0 is positioned at the origin by a translation. This establishes $(\nu_{0i}, \nu_{0k}) = (v_i, v_k)$, $1 \leqq i, k \leqq n, i \neq k$, as known information. Consequently, the inner product values of edge vectors co-original at v_1 are given by:

$$\begin{aligned}
(\nu_{1i}, \nu_{1k}) &= (v_i - v_1, v_k - v_1) \\
&= (v_1, v_1) - (v_1, v_k) - (v_i, v_1) + (v_i, v_k) \qquad 2 \leqq i, k \leqq n \quad i \neq k \\
(\nu_{10}, \nu_{1k}) &= (-v_1, v_k - v_1) \\
&= (v_1, v_1) - (v_1, v_k) \qquad 2 \leqq k \leqq n.
\end{aligned}$$

If the values (v_i, v_j), $i \neq j$, are constant, say c, then the above equalities give:

$$\begin{aligned}
(\nu_{1i}, \nu_{1k}) &= (v_1, v_1) - c \qquad 2 \leqq i, k \leqq n, i \neq k \\
(\nu_{10}, \nu_{1k}) &= (v_1, v_1) - c \qquad 2 \leqq k \leqq n.
\end{aligned}$$

Hence a constant value, c, at the vertex v_j establishes a constant value, $(v_i, v_i) - c$, at each of the other vertices v_i, $i \neq j$. In case $c \leqq 0$, then $(v_i, v_i) - c > 0$, $i \neq j$. Consequently at most one vertex can have a constant nonpositive value.

THEOREM 1. *An n-dimensional simplex may be placed with its vertices on the coordinate axes of E^{n+1}, if and only if all the triangle faces have acute interior angles, and at one vertex pairs of co-original edge vectors give equal inner products.*

Proof. Suppose that the $(n+1)$ vertices of an n-simplex are $a_0\phi_0, a_1\phi_1, \cdots, a_n\phi_n$, where again $\phi_0, \cdots, \phi_n$ are the standard orthonormal vectors, and $a_j \neq 0$, $j = 0, \cdots, n$. Then the n edge vectors, at say vertex v_0, are $a_1\phi_1 - a_0\phi_0, \cdots, a_n\phi_n - a_0\phi_0$, and we see that the inner product of any pair of these has value a_0^2. Similarly, for vertex v_i, the inner product of any pair of edge vectors is a_i^2. Thus a placeable n-simplex has a positive inner product value a_i^2 associated with each vertex v_i $(i = 0, \cdots, n)$, and in particular all angles between co-original edge vectors must be acute.

By Lemma 2, an n-simplex that satisfies the given conditions has a positive inner product value a_i^2 associated with each vertex v_i $(i = 0, 1, \cdots, n)$. The given simplex is congruent to the n-simplex described by the points

$$u_i = (\delta_{0i}a_0, \delta_{1i}a_1, \cdots, \delta_{ni}a_n)$$

$(i = 0, 1, \cdots, n)$. This follows from the previous Lemma 1 when we note that we have congruent facial triangles and consequently congruent side lengths in the proper correspondence.

It is interesting to note the following corollary. The proof is immediate from calculations arrived at after the tetrahedron ($n=3$) is placed. A similar relationship holds for nonadjacent edges of a placeable n-simplex for any $n>3$, but the related nonadjacent edges must be formed with end points of the original pair.

COROLLARY. *If a tetrahedron is placeable on the positive coordinate axes of E^4, then the sum of the squares of the lengths of two nonadjacent edges is equal to the sum of the squares of the lengths of any other two nonadjacent edges.*

The necessity of the acute angle hypothesis is demonstrated by the following example. Consider an n-simplex with the origin of E^n as one vertex and the other n vertices expressed as $v_i=(-a, -a, \cdots, -a, 1, -a, \cdots, -a)$, with the 1 in the ith position. Here the edge vectors co-original at the origin have pairwise inner products $(n-2)a^2-2a$, which are negative when $0<a<2/(n-2)$. Likewise the edge vectors co-original at v_i yield equal pairwise inner products $(1+a)^2>0$. Thus, we have an illustration of a nonplaceable simplex satisfying the equal inner product condition. The same example restricted to a tetrahedron shows that the corollary does not express a sufficient condition for placeability.

ON THE DIFFERENTIABILITY OF INDETERMINATE QUOTIENTS

LOWELL SCHOENFELD, Pennsylvania State University

The purpose of this paper is to prove the following result where N and D are complex-valued functions defined on a neighborhood of the real point ξ, and k and n are integers.

THEOREM. *Let $0\leqq k\leqq n$. Suppose that the derivatives $N^{(n)}$ and $D^{(n)}$ are continuous on a neighborhood of ξ, that $N^{(n+1)}(\xi)$ and $D^{(n+1)}(\xi)$ exist, and that*

$$(1)\qquad \begin{cases} N(\xi) = N'(\xi) = \cdots = N^{(k)}(\xi) = 0 \\ D(\xi) = D'(\xi) = \cdots = D^{(k)}(\xi) = 0 \neq D^{(k+1)}(\xi). \end{cases}$$

If f is defined by

$$f(x) = \begin{cases} N(x)/D(x) & \text{if } x \neq \xi \\ N^{(k+1)}(\xi)/D^{(k+1)}(\xi) & \text{if } x = \xi, \end{cases}$$

then $f^{(n-k)}$ exists and is continuous on some neighborhood of ξ.

Perhaps the most useful consequence is the following special case which arises on taking $n=1$ and $k=0$. A result of this kind which assumes, in addition, that N'' and D'' are continuous at ξ may be found in Rowland [2, Chapter 3, Section 3].

COROLLARY. *Suppose that N' and D' are continuous on a neighborhood of ξ, that $N''(\xi)$ and $D''(\xi)$ exist, and that $N(\xi)=D(\xi)=0\neq D'(\xi)$. If $f(\xi)=N'(\xi)/D'(\xi)$ and $f(x)=N(x)/D(x)$ for $x\neq\xi$, then f' exists and is continuous on a neighborhood of ξ.*

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Remark 1. Under the assumption that N and D are real-valued, that $N^{(k+1)}(\xi)$ and $D^{(k+1)}(\xi)$ exist and that (1) holds, it follows from l'Hospital's rule that $\lim_{x\to\xi}f(x)=f(\xi)$; thus our definition of $f(\xi)$ is the one required to ensure the continuity of f at ξ. Our result may therefore be considered to be one stating additional hypotheses which guarantee the continuity of $f^{(n-k)}$ rather than, merely, that of f itself.

Remark 2. If we drop the hypothesis that $N^{(n+1)}(\xi)$ exists, then $f^{(n-k)}(\xi)$ need not exist so that our result is sharp. To see this, take $n=1$ and $k=0$ so that we are concerned with the corollary. Let $\xi=0$, $N(0)=0$, $N(x)=x^{5/3}\sin x^{-1/3}$ if $x\neq 0$, and $D(x)=x$. Then all hypotheses are satisfied except that $N''(0)$ does not exist. Now $f(0)=0$ and $f(x)=x^{2/3}\sin x^{-1/3}$ for $x\neq 0$; hence $f'(0)$ does not exist.

Remark 3. If both N and D are analytic at ξ, then the conclusion readily follows from the factorizations

$$N(x) = (x-\xi)^{k+1}\nu(x), \qquad D(x) = (x-\xi)^{k+1}\delta(x)$$

where ν and δ are analytic at ξ and $\delta(\xi)\neq 0$. Our proof adapts this idea to non-analytic functions.

We give the proof after developing two lemmas. The first of these may be found in a different form in Rowland [2, Chapter 3, Section 4] and is also related to some work of Nitsche [1, Section 4].

LEMMA 1. *Let $L^{(n)}$ be continuous on a neighborhood of ξ and let $L^{(n+1)}(\xi)$ exist. Then there exists a function Ψ such that:*

$$\text{(a)} \qquad L(x) = \sum_{j=0}^{n+1}\frac{L^{(j)}(\xi)}{j!}(x-\xi)^j + \Psi(x),$$

(b) $\Psi^{(n)}$ *exists and is continuous on some neighborhood of* ξ,

(c) $\Psi^{(k)}(x) = o(x-\xi)^{n+1-k}$ *as* $x\to\xi$, $\quad 0\leqq k\leqq n$.

Proof. For $n=0$, the result is clear on defining $\Psi(x)=L(x)-L(\xi)-(x-\xi)L'(\xi)$. We therefore let $n\geqq 1$ and apply Taylor's Theorem to get

$$L(x) = \sum_{j=0}^{n-1}\frac{L^{(j)}(\xi)}{j!}(x-\xi)^j + \frac{1}{(n-1)!}\int_\xi^x (x-t)^{n-1}L^{(n)}(t)dt. \tag{2}$$

Let $\eta(\xi)=0$ and for $x\neq\xi$ let

$$\eta(x) = \frac{L^{(n)}(x)-L^{(n)}(\xi)}{x-\xi} - L^{(n+1)}(\xi);$$

then η is continuous on a neighborhood of ξ. Also

$$\begin{aligned}&\frac{1}{(n-1)!}\int_\xi^x (x-t)^{n-1}L^{(n)}(t)dt\\ &\quad = \frac{1}{(n-1)!}\int_\xi^x (x-t)^{n-1}\{L^{(n)}(\xi)+(t-\xi)L^{(n+1)}(\xi)+(t-\xi)\eta(t)\}dt\end{aligned}$$

$$(3) \qquad = \frac{L^{(n)}(\xi)}{n!}(x-\xi)^n + \frac{L^{(n+1)}(\xi)}{(n+1)!}(x-\xi)^{n+1} + \Psi(x)$$

where

$$\Psi(x) = \frac{1}{(n-1)!}\int_\xi^x (x-t)^{n-1}(t-\xi)\eta(t)dt.$$

On substituting (3) into (2) we get (a). Moreover, an induction on k easily establishes that

$$\Psi^{(k)}(x) = \frac{1}{(n-1-k)!}\int_\xi^x (x-t)^{n-1-k}(t-\xi)\eta(t)dt, \qquad 0 \leqq k \leqq n-1.$$

On taking $k=n-1$ and differentiating once more, we get $\Psi^{(n)}(x)=(x-\xi)\eta(x)$; hence (b) holds. Since $\eta(t)\to 0$ as $t\to\xi$, these results imply (c).

LEMMA 2. *Let $L^{(n)}$ be continuous on a neighborhood of ξ and let $L^{(n+1)}(\xi)$ exist. Let $0\leqq k\leqq n$, and define $\lambda(\xi)=L^{(k+1)}(\xi)/(k+1)!$ but for $x\neq\xi$ let*

$$\lambda(x) = \left\{L(x) - \sum_{j=0}^{k}\frac{L^{(j)}(\xi)}{j!}(x-\xi)^j\right\} \Big/ (x-\xi)^{k+1}.$$

Then $\lambda^{(n-k)}$ exists and is continuous on some neighborhood of ξ.

Proof. On applying Lemma 1, we obtain for $x\neq\xi$

$$(4) \qquad \lambda(x) = \sum_{j=k+1}^{n+1}\frac{L^{(j)}(\xi)}{j!}(x-\xi)^{j-1-k} + \phi(x)$$

where $\phi(x)=\Psi(x)(x-\xi)^{-k-1}$. If we define $\phi(\xi)=0$ then (4) holds for $x=\xi$ as well; also $\phi^{(n)}$ exists on a deleted neighborhood of ξ. Now for $x\neq\xi$ and $0\leqq m\leqq n-k$, we have by (c) that

$$\begin{aligned}\phi^{(m)}(x) &= \sum_{r=0}^{m}\binom{m}{r}\Psi^{(r)}(x)\cdot(-1)^{m-r}(x-\xi)^{-k-1-m+r}\\ &= \sum_{r=0}^{m}o(x-\xi)^{n+1-r}(x-\xi)^{-k-1-m+r} = o(x-\xi)^{n-k-m}\\ &= o(1) \qquad \text{as} \qquad x\to\xi.\end{aligned}$$

In particular, $\phi(x)\to 0$ as $x\to\xi$ so that ϕ is continuous on a neighborhood of ξ. Also $\phi'(x)\to 0$ as $x\to\xi$ so that $\phi'(\xi)$ exists and is 0; hence ϕ' is continuous on a neighborhood of ξ. An induction now establishes that for $0\leqq m\leqq n-k$ the function $\phi^{(m)}$ exists and is continuous on a neighborhood of ξ. On taking $m=n-k$ and using (4) we deduce the conclusion of the lemma.

Remark 4. From (a), it is easy to see that $\Psi^{(n+1)}(\xi)$ exists and is equal to 0. Moreover, if $0\leqq m\leqq n-k$ then $\phi^{(m)}(\xi)=0$, and (4) gives

$$\lambda^{(m)}(\xi) = \frac{m!}{(m+k+1)!}L^{(m+k+1)}(\xi).$$

Also Lemma 2 implies the main theorem for the case $D(x)=(x-\xi)^{k+1}$.

Proof of the theorem. Let the functions ν and δ correspond to N and D in the same way that λ corresponds to L. Hence $\delta(\xi)\neq 0$ because $D^{(k+1)}(\xi)\neq 0$; since δ is continuous at ξ, there is some neighborhood of ξ in which $\delta(x)$ is nowhere 0. By Lemma 2, $\nu^{(n-k)}$ and $\delta^{(n-k)}$ exist and are continuous on some neighborhood of ξ. For $x\neq\xi$, we have

$$f(x)=\frac{N(x)}{D(x)}=\frac{(x-\xi)^{k+1}\nu(x)}{(x-\xi)^{k+1}\delta(x)}=\frac{\nu(x)}{\delta(x)};$$

moreover, $f(x)=\nu(x)/\delta(x)$ holds for $x=\xi$ as well by our definitions of $f(\xi)$, $\nu(\xi)$, $\delta(\xi)$. Consequently, for all x in some neighborhood of ξ, there exists

$$f^{(n-k)}(x)=\frac{d^{n-k}}{dx^{n-k}}\left\{\frac{\nu(x)}{\delta(x)}\right\}=\frac{P(\nu,\nu',\cdots,\nu^{(n-k)},\delta,\delta',\cdots\delta^{(n-k)})}{\{\delta(x)\}^{n+1-k}}$$

where P is a polynomial in $2(n+1-k)$ variables. Hence $f^{(n-k)}$ is continuous on some neighborhood of ξ and the result is proved.

References

1. J. C. C. Nitsche, Über die Abhängigkeit der Tschebyscheffschen Approximierenden einer differenzierbaren Funktion von Intervall, Numer. Math., 4 (1962) 262–276.

2. John Rowland, On the location of the deviation points in Chebyshev approximation by polynomials, thesis, Pennsylvania State University, March, 1966.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

A Mathematician's Apology. By G. H. Hardy. Reprinted from the 1948 edition with a forward by C. P. Snow. Cambridge University Press, New York, 1967. 153 pp. $2.95.

This is the long awaited reissue of the *Apology* which first appeared in 1940. Hardy sets the theme of this exciting, beautifully written essay on the fifth page of the Apology. He writes: "I shall ask, then, why is it really worth while to make a serious study of mathematics? What is the proper justification of a mathematician's life? And my answers will be, for the most part, such as are to be expected from a mathematician . . . But I shall say at once that, in defending mathematics, I shall be defending myself, and that my apology is bound to be to some extent egotistical."

Hardy's defense of mathematics can be read by anyone; except for a theorem or two there is nothing here that the layman cannot understand. However, in

Also Lemma 2 implies the main theorem for the case $D(x)=(x-\xi)^{k+1}$.

Proof of the theorem. Let the functions ν and δ correspond to N and D in the same way that λ corresponds to L. Hence $\delta(\xi)\neq 0$ because $D^{(k+1)}(\xi)\neq 0$; since δ is continuous at ξ, there is some neighborhood of ξ in which $\delta(x)$ is nowhere 0. By Lemma 2, $\nu^{(n-k)}$ and $\delta^{(n-k)}$ exist and are continuous on some neighborhood of ξ. For $x\neq\xi$, we have

$$f(x)=\frac{N(x)}{D(x)}=\frac{(x-\xi)^{k+1}\nu(x)}{(x-\xi)^{k+1}\delta(x)}=\frac{\nu(x)}{\delta(x)};$$

moreover, $f(x)=\nu(x)/\delta(x)$ holds for $x=\xi$ as well by our definitions of $f(\xi)$, $\nu(\xi)$, $\delta(\xi)$. Consequently, for all x in some neighborhood of ξ, there exists

$$f^{(n-k)}(x)=\frac{d^{n-k}}{dx^{n-k}}\left\{\frac{\nu(x)}{\delta(x)}\right\}=\frac{P(\nu,\nu',\cdots,\nu^{(n-k)},\delta,\delta',\cdots\delta^{(n-k)})}{\{\delta(x)\}^{n+1-k}}$$

where P is a polynomial in $2(n+1-k)$ variables. Hence $f^{(n-k)}$ is continuous on some neighborhood of ξ and the result is proved.

References

1. J. C. C. Nitsche, Über die Abhängigkeit der Tschebyscheffschen Approximierenden einer differenzierbaren Funktion von Intervall, Numer. Math., 4 (1962) 262–276.

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Hardy's defense of mathematics can be read by anyone; except for a theorem or two there is nothing here that the layman cannot understand. However, in order to fully appreciate the *Apology* one should know something about Hardy as a man, and also have some knowledge of his mathematical achievements.

Thus the reviewer was extremely pleased to see a forward included in this edition which gives an excellent sketch of Hardy's life. C. P. Snow traces Hardy's life from his birth in Surrey in 1877 until his tragic death in 1947. We learn about his parents and about his early education. We follow his career as a Fellow at Trinity College, Cambridge (1898–1919); his stay at Oxford (1919–1931); and his return to Cambridge. Snow tells of the one romantic incident in Hardy's life, his discovery of Ramanujan, the poor Indian clerk born with a genius for mathematics.

From Snow's sketch Hardy emerges as a strange man, an eccentric who hated telephones and watches, and wouldn't have a looking-glass in his room; a man so shy that he dreaded introductions, but brave enough to stand up for his beliefs; a man with two strong passions, mathematics and cricket.

Hardy hated war and this perhaps explains to some extent why he regarded applied mathematics (ballistics, for example) as "repulsively ugly and intolerably dull." He writes: "I have never done anything 'useful'. No discovery of mine has made or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world." This, of course, is not true, for it has often been pointed out that some of Hardy's own pure mathematical discoveries have found applications in applied fields (see Newman, *The World of Mathematics*, p. 2026). This is one of the attractions of the *Apology*; the reader quickly finds himself in agreement with, or quite opposed to some of Hardy's statements.

Hardy was a pure mathematician whose main work was in analysis and the theory of numbers (or the higher arithmetic). His great reputation, of course, rests on his very original and advanced research papers, many of the most famous of which were written in collaboration with J. E. Littlewood, and with Ramanujan. The first volume of his *Collected Papers* has been published by the Oxford University Press. This is a volume of 700 pages; eventually there will be seven volumes.

Snow's moving, beautifully written story of Hardy's last years explains clearly why many regard the *Apology* as he does as a "book of haunting sadness." Hardy died December 1, 1947, the day the Copley Medal of the Royal Society, its highest award, was to have been presented to him.

There is a striking photograph of Hardy on the jacket of the book, and a sample of his writing style on the back. Since jackets get torn and lost, it is a pity the photo does not face the title page. Since he hated to be photographed there are few pictures of him.

We conclude this review by quoting from the *Scientific Monthly*: "Hardy's sardonic confession of how he ever came to be a professional mathematician may be specially recommended to solemn young men who believe they have the call to preach the higher arithmetic to mathematical infidels . . . "

C. D. OLDS, San Jose State College

Mathematics and Computing: with FORTRAN Programming. By W. S. Dorn and H. J. Greenberg. Wiley, New York, 1967. xvi+595 pp. $8.95.

The use of the computer offers the student a mathematical tool for problem solving that can help develop insight, and in fact allows him to make discoveries that otherwise might not be possible. Dorn and Greenberg have recognized this concept and have exploited it well. Their book supposes two or three years of high school mathematics as preparation. From there it presents at a modest pace the topics usually referred to as finite mathematics with an introduction to intuitive calculus. An introduction to matrices, a topic most frequently included, is missing here.

The central unifying tool is the computational aspect of the approach. It is an algorithmic approach and is carefully carried out. Each new topic is introduced by a well chosen example. Understanding is encouraged through many well graded exercises both in the body of the text and at the end of each chapter. Although the approach is intuitive, the importance of verifying one's results and proving theorems is frequently pointed out.

In Chapters 1 and 2, the solution of simultaneous linear equations and linear inequalities is presented with applications to linear programming. The flow chart is introduced and used effectively in the explanations. This is followed in Chapter 3 by a presentation of elementary FORTRAN programming sufficient for handling the problems introduced.

Chapters 4 through 7 introduce intuitive calculus. The idea of derivative is introduced by calculating the speed of a moving object. The power of the nonending algorithm is developed nicely through solving of nonlinear equations. This and the infinite sums in Chapter 6 give the student a good grasp of convergence as well as a means of estimating the accuracy of his results.

Chapter 8 introduces logic and Boolean Algebra. With the computer interesting examples in circuit analysis are carefully developed and carried a little farther than usual.

The final chapter is concerned with probability. I regret this important topic is presented last and perhaps more hastily treated as a result. In this 60 page chapter are introduced sample space, tree diagrams, compound probability, Markov processes, Monte Carlo methods and random number generators.

The authors suggest that the book may be used for a fourth year of mathematics in high school or a terminal course for the nonscience major in college. I feel that it touches briefly on too many topics to be easily handled by some high school teachers, especially those whose own training may not have included many of the applications and new approaches. I have chosen to use the book in an NSF institute for junior high school and high school teachers. In places, some supplementing seems essential. A brief introduction to program construction prior to the discussion involving subscripted variables would have been in order. In general, I found the approach invigorating, interesting to the student (in particular, the more mature student) and well carried out.

A. E. HALTEMAN, San Jose State College

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in india ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

To be considered for publication, solutions should be mailed before October 1, 1968.

PROPOSALS

691. *Proposed by Charles W. Trigg, San Diego, California.*

Using the nine positive digits just once each, form two integers A and B such that $A=8B$.

692. *Proposed by Michael J. Martino, Temple University.*

Prove that $1!+2!+3!+\cdots+k!$ is asymptotic to $(k+1)!/k$ as $k\to\infty$.

693. *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

A square sheet of one cycle by one cycle log log paper is ruled with n vertical lines and n horizontal lines. Find the number of perfect squares on this sheet of logarithmic graph paper.

694. *Proposed by J. S. Vigder, Defence Research Board of Canada, Ottawa, Canada.*

Find all the triangles with integral sides in which the area and perimeter are equal to the same integer.

695. *Proposed by John M. Howell, Los Angeles City College.*

A population consists of a items of type A and b items of type B. One at a time is drawn without replacement until all of the type A items are drawn. What is the probability that this happens on the xth draw where $x=a, a+1, \cdots, a+b$?

696. *Proposed by H. W. Vayo and R. W. Shoemaker, University of Toledo.*

Given a prolate spheroidal surface, if one rotates the transverse midsection about the minor axis, one obtains an elliptical section on the surface. Find an expression for the length of the semimajor axis of this ellipse in terms of the angle of rotation and also an expression for the eccentricity of the ellipse.

697. *Proposed by Erwin Just, Bronx Community College.*

Assume that each member of the sequence of functions $f_1, f_2, \cdots, f_n$ is differentiable on $[a, b]$, $f_1(a)=f_n(b)=0$ and $f_i(x)\neq 0$ when $x\in(a, b)$, $i=1, 2, \cdots, n$. Prove that there exists $c\in(a, b)$ such that $\sum_{i=1}^{n} f_i'(c)/f_i(c)=0$.

SOLUTIONS

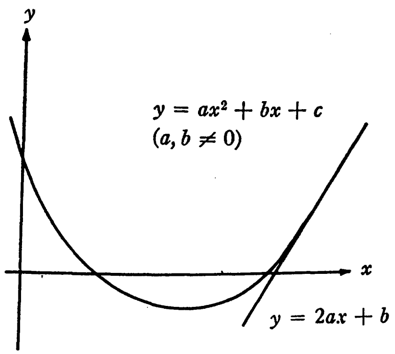
Late Solutions

John Beidler, University of Scranton: **664, 667**; *David Fettner, City College of New York:* **648**; *Richard A. Jacobson, Houghton College:* **655, 656**; *James G. Seiler, San Diego City College:* **663**; *and K. L. Singh, Memorial University of Newfoundland:* **632**.

The Impossible Figure

669. [November, 1967] *Proposed by Robert P. Baker, Newark, New Jersey.*

Why is the figure below impossible?



I. *Solution by Stephen Spindler, Purdue University.*

From the figure the polynomial ax^2+bx+c has two real roots; thus $b^2-4ac>0$, $2ax+b=0$ implies $x=-b/2a$; but the figure implies $(-b+\sqrt{b^2-4ac})/2a<-b/2a$, which is impossible.

II. *Solution by D. L. Muench, St. John Fisher College, New York.*

It is obvious that the x-intercept of the straight line is $-b/2a$ which is also the abscissa of the vertex of the parabola. In the given diagram, these two points do not coincide. Hence the contradiction.

Also solved by Philip Ancona, University of Michigan; M. S. Aurora and Merrill Barnebey (jointly), Wisconsin State University, LaCrosse; Leon Bankoff, Los Angeles, California; Donald Batman, M.I.T. Lincoln Laboratory; John Beidler, University of Scranton; Richard J. Bonneau, Holy Cross College; Wray G. Brady, University of Bridgeport; Dermott A. Breault, Harvard Computing Center; Robert J. Bridgman, Mansfield State College, Pennsylvania; Nicholas C. Bystrom, Northland College, Wisconsin; C. Robert Clements, Choate School, Connecticut; George F. Corliss, College of Wooster, Ohio; Mickey Dargitz, Ferris State College, Michigan (Two solutions); Edward T. Delaney, Seton Hall University; Alan Edmonds, Oklahoma State University; Mary J. Ellien, University of Connecticut; Sol L. Feigenbaum, West Haven, Connecticut; Neal Felsinger, Yale University; Martin Feuerman, USN Oceanographic Office, Maryland; Philip Fung, Cleveland State College; Arnold Funkenbusch, Houghton High School, Michigan; Hyman Gabai, York College, New York; Mrs. A. C. Garstang, Boulder, Colorado; Harry M. Gehman, SUNY at Buffalo, New York; Anton Glaser, Pennsylvania State University, Abington, Pennsylvania; Michael Goldberg, Washington, D.C.; H. S. Hahn, West Georgia College; Carl Hammer, UNIVAC, Washington, D.C.; J. Ray Hanna, University of Wyoming; Ned Harrell, Menlo-Atherton High School, California; Ruth E. Heintz, State University

College, Buffalo, New York; John O. Herzog, Pacific Lutheran University, Washington; Kent C. Iberg, Illinois State University; Richard A. Jacobson, Houghton College, New York; Gerald G. Jahn, Wisconsin State University, Eau Claire; Douglas H. Johnson, University of Wisconsin; Erwin Just, Bronx Community College; Carol Lynne Keller, Texas Technological College; Bruce W. King, Burnt Hills-Ballston Lake High School, New York; Lew Kowarski, Morgan State College, Maryland; Eric S. Langford, Naval Postgraduate School, California; David C. Lantz, Kutztown State College; J. F. Leetch, Bowling Green State University, Ohio; Peter A. Lundstrom, Union College, New York; John E. McDonald, Jr., Wappingers Falls, New York; Wilber H. McKenzie, City College of San Francisco; Otto Mond, White Plains, New York; John J. Moore; John K. Moulton, Wellesley Hills, Massachusetts; Joseph E. Mueller, Bloomsburg State College, Pennsylvania; Donald A. Myers, University of Iowa; C. Stanley Ogilvy, Hamilton College, New York; Don N. Page, William Jewell College, Missouri; Charles F. Pinzka, University of Cincinnati; Stephen K. Prothero, Willamette Univ., Oregon; Thomas J. Page, Bloomington, Illinois; Frank J. Papp, University of Delaware; Alan H. Price, Honolulu, Hawaii; Stanley Rabinowitz, Far Rockaway, New York; Mrs. Elaine Rakshys, Newton South High School, Massachusetts; Arnold Rais, University of Waterloo, Canada; Kenneth A. Ribet, Brown University; Henry J. Ricardo, Yeshiva University; Ellis J. Rich, SUNY Maritime College; Marilyn Rodeen, San Francisco, California; Nathan Rubinstein, Johns Hopkins University; Albert Schild, Temple University; Harry Siller, Hofstra University; E. P. Starke, Plainfield, New Jersey; Paul Sugarman, Swampscott, Massachusetts; Frederick W. Terry, II, Elizabethtown College, Pennsylvania; John Hudson Tiner, Harrisburg, Arkansas; Zalman Usiskin, University of Michigan; Dimitrios Vathis, Chalcis Greece; Julius Vogel, Newark, New Jersey; E. W. Wallace, University of Leeds, England; A. B. Western, Melbourne High School, Florida; Samuel Wolf, Lithicum Heights, Maryland; Frederick H. G. Wright II, Plymouth-Carrer High School, Massachusetts; Gregory Wulczyn, Bucknell University; K. L. Yocum, South Dakota State University; and the proposer.

A Christmas Cryptarithm

670. [November, 1967] *Proposed by Maxey Brooke, Sweeny, Texas.*

Find the unique solution: · · · nothing you dismay!

$$\begin{array}{r} MERRY \\ XMAS \\ FROM \\ \hline MAXEY \end{array}$$

I. *Solution by Lawrence V. Novak, Pennsylvania State University.*

If the phrase " . . . nothing you dismay" is interpreted to mean the admission of only nonzero integers, there are at least four solutions, as follows:

$$\begin{array}{r} 74665 \\ 1793 \\ 2687 \\ \hline 79145 \end{array} \qquad \begin{array}{r} 61779 \\ 2684 \\ 3756 \\ \hline 68219 \end{array} \qquad \begin{array}{r} 41337 \\ 2496 \\ 5384 \\ \hline 49217 \end{array} \qquad \begin{array}{r} 21557 \\ 4298 \\ 3562 \\ \hline 29417 \end{array}$$

If we allow zero, there are at least sixteen more solutions. Which just goes to show that there is no unique way of saying Merry Christmas!

II. *Solution by Carl Hammer, UNIVAC, Washington, D.C.*

The "unique" solution, obviously in the nonal system (A, E, F, M, O, R, S,

X, Y), is $E=0$ (" . . . *nothing* you dismay.") as determined by our UNIVAC 1107 in 256 seconds: this is the o n l y element in this cryptarithm which is common to all of its five solutions:

40223	30774	20335	10445	20446
1485	1386	1267	2168	3257
6274	5723	4382	3471	1482
48103	38104	26105	16205	25306

Of course, there exist also numerous decimal solutions as well as solutions in systems having bases greater than 10; their number grows rapidly as we increase the number system base and consequently allow additional degrees of freedom.

One or more solutions were obtained by Miguel Bamberger, University of New Mexico; Merrill Barnebey, Wisconsin State University, LaCrosse; Keith Boisvert, Loyola High School, Baltimore; Dermott A. Breault, Harvard Computing Center; C. Robert Clements, The Choate School, Connecticut; George F. Corliss, College of Wooster, Ohio; Octario Garcia, Instituto Technologico de Monterrey, Mexico; Anton Glaser, Pennsylvania State University, Abington, Pennsylvania; H. S. Hahn, West Georgia College; Regina R. Hoelscher, Colorado Springs, Colorado; J. A. H. Hunter, Toronto, Ontario, Canada; Richard A. Jacobson, Houghton College; Daniel Jandinski, Northampton High School, Massachusetts; Edgar Karst, University of Arizona; Lew Kowarski, Morgan State College, Maryland; Mark I. Krusemeyer, Soest, The Netherlands; David C. Lantz, Kutztown State College; James Lyness, Loyola High School, Towson, Maryland; J. J. Martinez, Loyola High School, Towson, Maryland; John W. Milsom, Butler County Community College, Pennsylvania; Otto Mond, White Plains, New York; John K. Muth, St. Timothy's School, Maryland; Don N. Page, William Jewell College, Missouri; Paul Sugarman, Swampscott, Massachusetts; Samuel Wolf, Linthicum Heights, Maryland; K. L. Yocom, South Dakota State University; and the proposer.

A Sum of Products

671. [November, 1967] *Proposed by A. Wilansky, Lehigh University.*

Let $t_1, t_2, \cdots, t_r$ be real numbers with $t_1+t_2+\cdots+t_r=0$ and let $\{x_n\}$ be a bounded sequence of real numbers. Show that if the $\lim_{n\to\infty}(t_1x_{n-r+1}+t_2x_{n-r+2}+\cdots+t_rx_n)$ exists, it must be zero.

I. *Solution by A. B. Western, Melbourne High School, Florida.*

Let $S_n=t_1x_{n-r+1}+t_2x_{n-r+2}+\cdots+t_rx_n$.
If $\lim_{n\to\infty} s_n$ exists, call it L, then

$$L-\epsilon<S_n<L+\epsilon,\quad \text{for all } n>N$$

Since $\{x_n\}$ is bounded it has a least upper bound and greatest lower bound, $\overline{X}$ and $\underline{X}$ respectively.

Consequently

$$L-\epsilon<S_n\leqq t_1\overline{X}+t_2\overline{X}+\cdots t_n\overline{X}=0$$

$$L+\epsilon>S_n\geqq t_1\underline{X}+t_2\underline{X}+\cdots t_n\underline{X}=0$$

Clearly L must be zero.

II. *Solution by Donald Batman, M.I.T. Lincoln Laboratory.*

Consider the following array:

$$\begin{array}{llllllll}
t_1x_{n-r+1} + & \cdots & & & + & t_rx_n & = & S_n \\
t_1x_{n-r+2} + & \cdots & + & t_{r-1}x_n & + & t_rx_{n+1} & = & S_{n+1} \\
\vdots & & & & & \vdots & & \vdots \\
t_1x_n & + & t_2x_{n+1} & + & \cdots & + \; t_rx_{n+r-1} & = & S_{n+r-1} \\
t_1x_{n+1} & + & t_2x_{n+2} & + & \cdots & + \; t_rx_{n+r} & = & S_{n+r}
\end{array}$$

Note that if we form the sum $S_n + \cdots + S_{n+r}$ the sums of the elements across the diagonals as indicated vanish. That is $t_1x_n + \cdots t_rx_n = x_n(t_1 + \cdots t_r) = 0$. This $S_n + S_{n+1} + \cdots + S_{n+k} = \overline{\Delta}_n + \underline{\Delta}_{n+k}$, where $\overline{\Delta}_n$ is the triangular sum in the upper left-hand corner and $\underline{\Delta}_{n+k}$ is the triangular sum in the lower right-hand corner.

Note that as we let k increase the number of elements in the sum $\underline{\Delta}_{n+k}$ does not increase whereas if the limit $\text{lin}_{n\to\infty}\,(t_1x_{n-r+1} + \cdots + t_rx_n)$ exists and is non-zero, the magnitude of the sum $S_n + S_{n+1} + \cdots + S_{n+k}$ increases without bound; hence the magnitudes of the elements included in the sum $\underline{\Delta}_{n+k}$ must likewise increase without bound contradicting the boundedness of $\{x_n\}$. The stated result therefore follows.

Also solved by John Beidler, University of Scranton, Pennsylvania; H. S. Hahn, West Georgia College; Carl Hammer, UNIVAC, Washington, D.C.; Richard A. Jacobson, Houghton College, New York; J. F. Leetch, Bowling Green State University, Ohio; R. S. Luthar, University of Wisconsin, Waukesha; and K. L. Yocom, South Dakota State University.

A Diophantine Quadratic

672. [November, 1967] *Proposed by R. S. Luthar, Colby College, Maine.*

Prove the theorem: *A sufficient condition that the equation $x^2 - dy^2 = -I$ be soluble in integers is that d be of the form $I + n^2$ or $5 + 4n(n+I)$ where n is any integer.*

Solution by J. A. H. Hunter, Toronto, Ontario, Canada.

If $d = I + n^2$, there will always be integral solutions from: $x = n$, $y = 1$. This part of the theorem is therefore proven.

If $d = 5 + 4n(n+I)$, however, we have a simple counterexample which invalidates the theorem:

$$\textit{Say } d = 17 = 5 + 4\cdot 1\cdot 3, \textit{ with } n = 1, I = 2.$$

Then we have $x^2 - 17y^2 = -2$, which can have no solution in integers. The second part of the theorem is therefore not valid.

E. P. Starke showed that in the case $I = 1$, the theorem holds for both conditions on d. He noted that the conditions are not necessary. For example, for $d = 41$, (32, 5) is a solution.

Also solved by Stanley Rabinowitz, Far Rockaway, New York and Kenneth M. Wilke, Topeka, Kansas.

Power of a Matrix

673. [November, 1967] *Proposed by Erwin Just, Bronx Community College.*

If n is an integer such that

$$\begin{bmatrix} k & m \\ 0 & k+1 \end{bmatrix}^n = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \qquad \text{prove that } a_2 = m[(k+1)^n - k^n].$$

I. *Solution by David Gootkind, Avco MSD, Massachusetts.*

We will show that

$$(1) \qquad \begin{bmatrix} k & m \\ 0 & k+1 \end{bmatrix}^n = \begin{bmatrix} k^n & m[(k+1)^n - k^n] \\ 0 & (k+1)^n \end{bmatrix}$$

by induction.

Assume the validity of equation (1) for some n.

$$\begin{aligned} \begin{bmatrix} k & m \\ 0 & k+1 \end{bmatrix}^{n+1} &= \begin{bmatrix} k^n m[(k+1)^n - k^n] \\ 0 \quad (k+1)^n \end{bmatrix} \begin{bmatrix} k & m \\ 0 & k+1 \end{bmatrix} \\ &= \begin{bmatrix} k^{n+1} & mk^n + m(k+1)[(k+1)^n - k^n] \\ 0 & (k+1)^{n+1} \end{bmatrix} \\ &= \begin{bmatrix} k^{n+1} & m[(k+1)^{n+1} - k^{n+1}] \\ 0 & (k+1)^{n+1} \end{bmatrix} \end{aligned}$$

Equation (1) is clearly valid for $n=1$.

II. *Solution by Kenneth A. Ribet, Brown University.*

We have

$$\begin{aligned} \begin{bmatrix} k & m \\ 0 & k+1 \end{bmatrix} &= k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & m \\ 0 & 1 \end{bmatrix} \\ &= k + A \text{ where } A \text{ is an idempotent matrix.} \end{aligned}$$

Hence

$$\begin{aligned} \begin{bmatrix} k & m \\ 0 & k+1 \end{bmatrix}^n &= \sum_{i=0}^{n} \binom{n}{i} k^{n-i} A^i \\ &= \sum_{i=0}^{n-1} \binom{n}{i} k^i A + k^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

which gives

$$\begin{aligned} a_i &= m \sum_{i=0}^{n-1} \binom{n}{i} k^i \\ &= m\left[\sum_{i=1}^{n} \binom{n}{i} k^i - k^n\right] \\ &= m[(k+1)^n - k^n] \end{aligned}$$

III. *Solution by J. R. Kuttler, Johns Hopkins University, Applied Physics Laboratory.*

Since

$$\begin{bmatrix} k & m \\ 0 & k+1 \end{bmatrix} = M \begin{bmatrix} k & 0 \\ 0 & k+1 \end{bmatrix} M^{-1},$$

where

$$M = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix},$$

it follows that

$$\begin{bmatrix} k & m \\ 0 & k+1 \end{bmatrix}^n = M \begin{bmatrix} k^n & 0 \\ 0 & (k+1)^n \end{bmatrix} M^{-1}$$

$$= \begin{bmatrix} k^n & m[(k+1)^n - k^n] \\ 0 & (k+1)^n \end{bmatrix}.$$

Also solved by Philip Ancona, University of Michigan; Merrill Barnebey, Wisconsin State University, LaCrosse; Donald Batman, M.I.T. Lincoln Laboratory; Dermott A. Breault, Harvard Computer Center; Wray G. Brady, University of Bridgeport; Nicholas C. Bystrom, Northland College, Wisconsin; Red Cougar, University of Houston; Mickey Dargitz, Ferris State College, Michigan; Neal Felsinger, Yale University; A. M. Fink, Iowa State University; Philip Fung, Cleveland State University; Mrs. A. C. Garstang, Boulder, Colorado; Michael Goldberg, Washington, D.C.; H. S. Hahn, West Georgia College; Carl Hammer, UNIVAC, Washington, D.C.; J. Ray Hanna, University of Wyoming; John O. Herzog, Pacific Lutheran University, Wash.; John M. Howell, Los Angeles City College; Walt Hillman, California State Polytechnic College, Pomona; Richard A. Jacobson, Houghton College, New York; Douglas H. Johnson, University of Wisconsin; Bruce W. King, Ballston Lake, New York; Lew Kowarski, Morgan State College, Maryland; Eric S. Langford, Naval Postgraduate School, California; David C. Lantz, Kutztown State College; Herbert R. Leifer, Pittsburgh, Pennsylvania; Peter A. Lindstrom, Uniçn College, New York; Otto Mond, White Plains, New York; John J. Moore; D. L. Muench, St. John Fisher College, New York; F. D. Parker, St. Lawrence University; Edgar M. Pass, Atlanta, Georgia; Charles F. Pinzka, University of Cincinnati; William G. Poole, Jr., University of California, Berkeley; Phyllis E. Pusey, University of North Carolina, Greensboro; Ray Van Raamsdonk, University of Alberta; Stanley Rabinowitz, Far Rockaway, New York; Arnold Rais, University of Waterloo, Canada; Henry J. Ricardo, Yeshiva University; Nathan Rubinstein, Johns Hopkins University; James G. Seiler, San Diego City College; B. T. Sims, Eastern Washington State College; Stephen Spindler, Purdue University; Zalman Usiskin, University of Michigan; Julius Vogel, Newark, New Jersey; Howard L. Walton, Falls Church, Virginia; A. B. Western, Melbourne High School, Florida; Kenneth M. Wilkie, Topeka, Kansas; Gregory Wulczyn, Bucknell University; K. L. Yocom, South Dakota State University; Eugene J. Zirkel, Nassau Community College, New York; and the proposer.

A False Conjecture

674. [November, 1967] *Proposed by J. A. H. Hunter, Toronto, Canada.*

Max Rummey (London, England) recently conjectured that $6^n/2 \pm 1$ must generate at least one prime for all values of n. Prove or disprove this conjecture.

Solution by Zalman Usiskin, University of Michigan.

We consider the primes in order. None of the numbers in question is divisible by 7, for $3\cdot 6^{n-1}\equiv \pm 3 \pmod 7$. Continuing,

$$\frac{6^n}{2}\equiv 1 \pmod{11} \Leftrightarrow n\equiv 9 \pmod{10}$$

$$\frac{6^n}{2}\equiv -1 \pmod{13} \Leftrightarrow n\equiv 11 \pmod{12}$$

The solution of this system is $n\equiv 59 \pmod{60}$. Thus $(6^{59}/2)-1$ is divisible by 11, $(6^{59}/2)+1$ is divisible by 13, and the conjecture is false.

By solving similar systems, the smallest counterexample I could find was for $n=19$, for 67 divides $(6^{19}/2)+1$, 11 divides $(6^{19}/2)-1$.

Also solved by Neal Felsinger, Yale University; H. S. Hahn, West Georgia College; Burrell W. Helton, Southwest Texas State College; Erwin Just, Bronx Community College; Edgar Karst, University of Arizona; Sidney Kravitz, Dover, New Jersey; Prasert Na Nagara, Kasetsart University, Thailand; Frank J. Papp. University of Delaware; Stanley Rabinowitz, Far Rockaway, New York; E. P. Starke, Plainfield, New Jersey; Gregory Wulczyn, Bucknell University; and the proposer.

A Well-Known Construction

675. [November, 1967] *Proposed by Philip Fung, Cleveland State University, Ohio.*

It is well known that in an obtuse triangle, the minimum number of lines necessary for partitioning into acute triangles is seven. Show a constructional method for such a partition.

Solution by Michael Goldberg, Washington, D.C.

The solution, published in *The American Mathematical Monthly*, 67 (1960) 923, in answer to my proposal, describes the following satisfactory dissection of the triangle ABC, where A is the obtuse angle. Draw the lines DE and FG (D on AB, G on AC, E and F on BC) tangent to the inscribed circle of center O, so that DE is perpendicular to OB, and FG is perpendicular to OC. Then the triangles BDE and FGC are acute isosceles triangles. All the angles of the pentagon $ADEFG$ are obtuse. Draw the lines OA, OD, OE, OF, OG which bisect the angles of the pentagon and divide the pentagon into five triangles. The central angles at O are all acute since each of the other angles of these five triangles are greater than 45°. Hence, all seven triangles are acute.

Also solved by Neal Felsinger, Yale University; H. S. Hahn, West Georgia College; Richard A. Jacobson, Houghton College, New York; Charles F. Pinzka, University of Cincinnati; Frederick H. G. Wright II, Plymouth-Carver High School, Massachusetts; and the proposer.

A Factorial Quotient

676. [November, 1967] *Proposed by William Squire, West Virginia University.*

Evaluate $\sum_{k=1}^{\infty} A_k^3$ where A_k is the factorial quotient

$$\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}.$$

Solution by H. W. Gould, West Virginia University.

The solution to this problem is obtained by way of Problem 612 (see comment on Pages 52 and 53, this MAGAZINE, January, 1967). The sum S there is the same as this problem save for the first term, omitted in the present problem. Indeed this follows because of the fact that

$$\binom{2n}{n}/2^{2n} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}.$$

Thus if $F=\sum_{k=1}^{\infty} A_k^3$ and as $S=\sum_{k=0}^{\infty} A_k^3$ was evaluated as $(4\pi^3)^{-1}\Gamma(1/4)^4$ we have $F=S-1=0.393203929685+$.

Also solved by Robert J. Bridgman, Mansfield State College, Pennsylvania; David Gootkind, Avco MSD, Massachusetts; and the proposer. One incorrect solution was received.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q430. Find a "simple" nth term formula for the sequence 0, 1, -1, 0, 0, -1, 1, 0, 0, 1, -1, 0, 0, -1, 1, 0, 0, 1, $-1, \cdots$.

[*Submitted by Murray S. Klamkin*]

Q431. Construct an orbit of a spaceship such that the moon and earth will always appear equally large to the astronauts.

[*Submitted by Charles E. Maley*]

Q432. Prove that each median of a triangle is shorter than the average of the adjacent sides.

[*Submitted by David L. Silverman*]

Q433. Prove the following inequalities:

$$\frac{\log_b (n-1)}{\log_b e} < \sum_{k=1}^{n} 1/k \leqq \frac{\log_b (n+1)}{\log_b 2}$$

where $b>0$, $\neq 1$.

[*Submitted by Norman Schaumberger*]

Q434. Does there exist a real number A such that $\lim_{x\to\infty}(\sqrt{x^2+x+1}-Ax)$ exists?

If so, what is the limit?

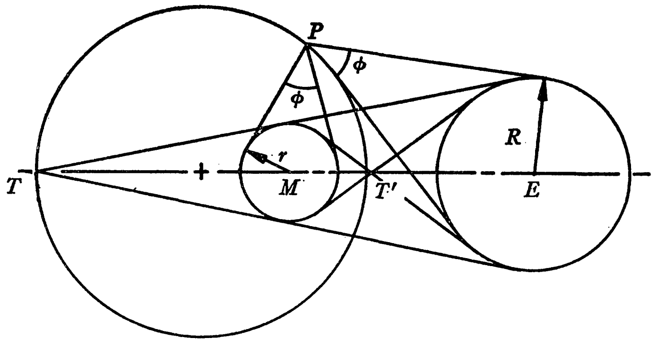
[*Submitted by E. M. Pass*]

(Answers on page 133)

ANSWERS

A430. One possible answer is: $\sin \pi(n^2-n)/4$.

A431.



Since subtended angles ϕ must be equal, two points are T and T', intersections of the pairs of common external and internal tangents respectively. From $ctn^2\phi/2=(PE^2-R^2)/R^2=(PM^2-r^2)/r^2=$relative power, all the points form a circle centered on the line of centers, $TMT'E$, that is, the circle of similitude.

A432. By considering the parallelogram formed by reflecting the triangle through the midpoint of the opposite side, the proposition becomes equivalent to the triangle inequality.

A433. Since $2\leqq(1+1/k)^k<e$, it follows that

$$\sum_{k=1}^{n} 1/k \log_b 2 \leqq \sum_{k=1}^{n} 1/k \log_b (1+1/k)^k < \sum_{k=1}^{n} 1/k \log_b e \tag{1}$$

Using

$$\log_b (n+1) = \log_b \left[\prod_{k=1}^{n} (k+1)/k \right]$$
$$= \sum_{k=1}^{n} 1/k \log_b (1+1/k)^k$$

we replace the middle term in (1) by $\log_b(n+1)$ and obtain an expression equivalent to the desired result.

A434. Let $u=1/x$. Then

$$\lim_{x\to\infty} (\sqrt{x^2+x+1}-Ax) = \lim_{u\to 0^+} \left(\frac{\sqrt{1+u+u^2}-A}{u}\right)$$

This can only exist if

$$A = \lim_{u\to 0^+} \sqrt{1+u+u^2} = 1.$$

Applying l'Hospital's rule

$$\lim_{u\to 0^+}\left(\frac{\sqrt{1+u+u^2}-1}{u}\right)=\lim_{u\to 0^+}\left(\frac{1+2u}{2\sqrt{1+u+u^2}}\right)=1/2.$$

(Quickies on page 166)

ON CURVES WITH CORNERS

JOHN H. STAIB, Drexel Institute of Technology

In an introduction to the subject of line integrals it is usually thought appropriate to simplify matters by restricting attention to a class of curves that is "sufficiently general for most applications." Most authors require that the parametric representation of the curve, say $P(t)$, $t\in[a, b]$, should be "piecewise smooth." An appropriate analytic definition for this term is given and it is indicated that the curve traversed by $P(t)$ will be "visually smooth" except, perhaps, for a finite number of "corners." It seems not to be noted, however, that such curves can be described by traversals that are continuously differentiable throughout their domain. This is suggested by the following example.

Example 1. Let

$$x(t)=\begin{cases}-t^2, & t<0\\ 0, & t\geqq 0\end{cases}\quad\text{and}\quad y(t)=\begin{cases}0, & t<0\\ t^2, & t\geqq 0.\end{cases}$$

Then the traversal $P(t)=(x(t), y(t))$, $t\in(-\infty, \infty)$, describes a curve C that has a 90° turn. (See Figure 1.) And yet $P(t)$ is continuously differentiable.

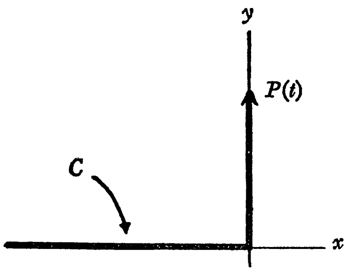


FIG. 1.

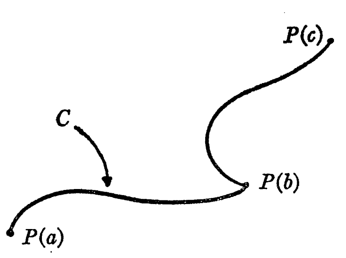


FIG. 2.

There is a "trick" to getting $P(t)$ around a corner without disturbing the continuity of its derivative: we must permit $P(t)$ to slow down to a halt as it rounds the corner. But this trick is always available. For suppose that a given $P(t)$ traverses the curve C shown in Figure 2 and that $P(t)$ is continuously differentiable except at $t=b$. Then we have only to replace $P(t)$ by a second traversal of C, say $Q(u)$, that has the special property that $Q'(\beta)=0$, where $Q(\beta)=P(b)$.